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Alexander Isaev

Lectures on the Automorphism Groups of Kobayashi-Hyperbolic Manifolds

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Preface

This book is based on a series of lectures that I gave at the Geometry and Analysis Seminar held in the Mathematical Sciences Institute of the Australian National University in October–November, 2005. For some time now I have been interested in characterizations of complex manifolds by their holomorphic automorphism groups, and my lectures summarized the results that I obtained in this direction (on some occasions jointly with S. Krantz and N. Kruzhilin) for the class of Kobayashi-hyperbolic manifolds during 2000–2005, with the majority of results produced in 2004–2005.

Here I give a coherent exposition (that includes complete proofs) of results describing hyperbolic manifolds for which the automorphism group dimensions are "sufficiently high" (this will be made precise in Chap. 1). The classification problem for hyperbolic manifolds with high-dimensional automorphism group can be thought of as a complex-geometric analogue of that for Riemannian manifolds with high-dimensional isometry group, which inspired many results in the 1950's-70's. Although the methods presented in the book are almost entirely different from those utilized in the Riemannian case, there is a common property that made these classifications possible: both the action of the holomorphic automorphism group of a hyperbolic manifold and the action of the isometry group of a Riemannian manifold are proper on the respective manifolds.

The book is organized as follows. In Chap. 1 I give a precise formulation of the classification problem that will be solved and summarize the main tools used in our arguments throughout the book. The classification problem splits into the homogeneous and non-homogeneous cases. The homogeneous case is treated in Chap. 2, whereas the more interesting non-homogeneous case from which the majority of manifolds arise occupies Chaps. 3–5. The general scheme for classifying non-homogeneous manifolds is, firstly, to describe all possible orbits (they turn out to have codimension 1 or 2) and, secondly, to study ways in which they can be joined together to form a hyperbolic manifold. The most challenging part of the arguments is dealing with Levi-flat and codimension 2 orbits, which becomes especially involved for 2-dimensional hyperbolic manifolds with 3-dimensional automorphism group considered in Chap. 5.

The book concludes with a discussion of possible generalizations of the classification results to the case of not necessarily hyperbolic complex manifolds that admit proper effective actions by holomorphic transformations of Lie groups of sufficiently high dimensions (see Chap. 6).

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Introduction

1.1 The Automorphism Group as a Lie Group

For a locally connected locally compact Hausdorff topological space X, let $\mathcal{H}(X)$ be the group of homeomorphisms of X. We equip $\mathcal{H}(X)$ with the *compact-open topology* which is the topology with subbase given by all sets of the form

$$\{f \in \mathcal{H}(X) : f(K) \subset U\},$$

where $K \subset X$ is compact and $U \subset X$ is open. It is well-known that $\mathcal{H}(X)$ is a topological group in this topology (see [Ar]). Every subgroup $G \subset \mathcal{H}(X)$ is a topological group in the topology induced from that of $\mathcal{H}(X)$; we call it the *compact-open topology on* G.

Let now M be a connected complex manifold, and $\mathrm{Aut}(M)$ the group of holomorphic automorphisms of M. Our immediate goal is to identify situations when one can introduce on $\mathrm{Aut}(M)$ a Lie group structure in the compact-open topology. Due to classical results of Bochner and Montgomery, the group $\mathrm{Aut}(M)$ is a Lie transformation group whenever it is locally compact (see [BM1], [BM2], [MZ]). We will now formulate a sufficient condition for the local compactness of general topological groups acting on manifolds.

A topological group G is said to *act continuously by diffeomorphisms* on a smooth manifold N, if a continuous homomorphism $\Phi : G \to \mathrm{Diff}(N)$ is specified, where $\mathrm{Diff}(N)$ is the group of all diffeomorphisms of N onto itself equipped with the compact-open topology. The continuity of Φ is equivalent to the continuity of the *action map*

$$G \times N \to N, \quad (g, p) \mapsto gp,$$

where $gp := \Phi(g)(p)$ (see [Du]). We will only be interested in *effective* actions, i.e. actions for which Φ is injective. The action of G on N is called *proper*, if the map

$$\Psi : G \times N \to N \times N, \quad (g, p) \mapsto (gp, p),$$

is proper, i.e. for every compact subset $C \subset N \times N$ its inverse image $\Psi^{-1}(C) \subset G \times N$ is compact as well. In the rich theory of Lie group actions on manifolds, work has been traditionally focused on compact groups. However, already the work of Palais (see [Pa]) on the existence of slices drew attention to the much larger class of proper actions that turned out to possess many of the good properties of compact group actions. Proper actions will be central to our exposition. For a complex manifold M we will be interested in proper actions *by holomorphic transformations* on M, in which case the group $\mathrm{Diff}(M)$ is replaced with the smaller group $\mathrm{Aut}(M)$. We will formulate and utilize some of the properties of proper actions in later sections. Here we only note that it follows directly from the definition above (see e.g. [Ak]) that a group acting properly on a manifold is locally compact. More information on proper group actions can be found in [Bi], [DK].

It was shown by Kaup in [Ka] that the $\mathrm{Aut}(M)$-action on M is proper if M admits a continuous $\mathrm{Aut}(M)$-invariant distance function. We will now describe a large class of manifolds with this property. First, on every connected complex manifold we introduce a pseudodistance known as the *Kobayashi pseudodistance* (see [Ko1]). Let $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in $\mathbb{C}$. The *Poincaré distance* on Δ is defined as

$$\rho(p,q) := \frac{1}{2} \ln \frac{1 + \left| \dfrac{p-q}{1-\overline{p}q} \right|}{1 - \left| \dfrac{p-q}{1-\overline{p}q} \right|},$$

where $p, q \in \Delta$. The Kobayashi pseudodistance on M can now be introduced as follows:

$$K_M(p,q) = \inf \sum_{j=1}^{m} \rho(s_j, t_j),$$

for all $p, q \in M$, where the inf is taken over all $m \in \mathbb{N}$, all pairs of points $\{s_j, t_j\}_{j=1,\dots,m}$ in Δ and all collections of holomorphic maps $\{f_j\}_{j=1,\dots,m}$ from Δ into M such that $f_1(s_1) = p$, $f_m(t_m) = q$, and $f_j(t_j) = f_{j+1}(s_{j+1})$ for $j = 1, \dots, m-1$. It is straightforward to verify that K_M is a pseudodistance on M, and one can easily think of examples when K_M is not a distance, that is, when $K_M(p,q) = 0$ for some $p, q \in M$, $p \neq q$. The Kobayashi pseudodistance does not increase under holomorphic maps, i.e. for any holomorphic map f between two complex manifolds M_1 and M_2 we have

$$K_{M_2}(f(p), f(q)) \leq K_{M_1}(p,q), \tag{1.1}$$

for all $p, q \in M_1$. In particular, K_M is $\mathrm{Aut}(M)$-invariant.

A complex manifold M for which the pseudodistance K_M is a distance function is called *Kobayashi-hyperbolic* or simply *hyperbolic*. For every manifold, K_M is continuous on $M \times M$ and, if M is hyperbolic, then K_M induces the topology of M (see [Roy], [Ba]). It now follows from the discussion above that if M is hyperbolic, then $\mathrm{Aut}(M)$ is a Lie group in the compact-open topology.

The class of hyperbolic manifolds is quite large. In fact, it cannot be expanded by means of considering other pseudodistances that do not increase under holomorphic maps. Indeed, the Kobayashi pseudodistance possesses a maximality property that we will now formulate. We say that an assignment of a pseudodistance d_M to every connected complex manifold M is a *Schwarz-Pick system* if:

(i) $d_\Delta = \rho$;

(ii) for every holomorphic map f between two complex manifolds M_1 and M_2 we have

$$d_{M_2}(f(p), f(q)) \le d_{M_1}(p, q),$$

for all $p, q \in M_1$.

Clearly, d_M is $\mathrm{Aut}(M)$-invariant for every M. It turns out that $\{K_M\}$ is the largest Schwarz-Pick system, that is, $K_{M_0}(p,q) \ge d_{M_0}(p,q)$ for every complex manifold M_0, every $p, q \in M_0$ and every Schwarz-Pick system $\{d_M\}$ (see e.g. [Di]). In particular, if for some Schwarz-Pick system $\{d_M\}$ and some manifold M_0 the pseudodistance d_{M_0} is in fact a distance, then M_0 is hyperbolic.

Consider, for example, for every connected complex manifold M, the classical *Carathéodory pseudodistance* defined as follows:

$$C_M(p,q) := \sup_f \rho(f(p), f(q)),$$

where $p, q \in M$ and the sup is taken over all holomorphic maps $f : M \to \Delta$. It is easy to show that $\{C_M\}$ is a Schwarz-Pick system (in fact, it is the smallest Schwarz-Pick system, that is, $C_{M_0}(p,q) \le d_{M_0}(p,q)$ for every complex manifold M_0, every $p, q \in M_0$ and every Schwarz-Pick system $\{d_M\}$ (see e.g. [Di])). Clearly, C_M is a distance function on every M for which points can be separated by bounded holomorphic functions (for example, on bounded domains in complex space). Hence all such manifolds are hyperbolic. There are, however, many more hyperbolic manifolds than those on which the Carathéodory pseudodistance is non-degenerate. For example, every compact complex manifold M not containing entire curves (that is, for which every holomorphic map from $\mathbb{C}$ into M is constant) is hyperbolic (see [Br]), while $C_M \equiv 0$ for compact M.

Apart from pseudodistances associated with Schwarz-Pick systems, there are distance functions invariant under biholomorphic maps but not necessarily non-increasing under arbitrary holomorphic maps. Such distances arise, for example, from the Bergman and Kähler-Einstein Hermitian metrics. The interested reader will find comprehensive treatments of the subject of invariant (pseudo)distances and (pseudo)metrics in [Di], [FV], [IKra1], [JP], [Ko1], [PoS], [Wu].

From now on M will denote a connected hyperbolic manifold. For such a manifold, we will work with the group $G(M) := \text{Aut}(M)^0$, the connected component of the identity of $\text{Aut}(M)$ (generally, for any topological group G, we denote by G^0 the connected component of the identity of G). Our arguments throughout the book heavily rely on the fact that the action of $G(M)$ is proper on M. In fact, most of our results can be generalized to the case of complex manifolds that admit proper effective actions by holomorphic transformations of certain groups (see Chap. 6).

1.2 The Classification Problem

The ultimate goal of this book is to obtain a complete classification of hyperbolic manifolds with "large" automorphism group, where we say that the group is large, whenever its dimension $d(M) := \dim \text{Aut}(M)$ is sufficiently high. Before formulating the classification problem precisely, we remark that it is reminiscent of that for Riemannian manifolds with high-dimensional isometry group. It was shown in [MS] that, for a smooth Riemannian manifold N, the group $\text{Isom}(N)$ of all isometries of N is a Lie group in the compact-open topology. Indeed, it was proved in [MS] that $\text{Isom}(N)$ coincides with the group $\text{Isom}_d(N)$ of all isometries of N regarded as a metric space with the distance function d arising from the Riemannian metric. The distance d induces the topology of N and thus turns N into a locally compact metric space. It then follows from [vDvdW] that the group $\text{Isom}_d(N)$ acts properly on N and therefore, arguing as in the previous section, we obtain that $\text{Isom}(N)$ is a Lie group (note, however, that the original argument in [MS] was different since the results of Bochner and Montgomery were not available at the time). By using the properness of the $\text{Isom}(N)$-action on N, it is straightforward to show that $\dim \text{Isom}(N) \le m(m+1)/2$, where $m := \dim N$. Furthermore, it turns out that the maximal value $m(m+1)/2$ is realized only for a manifold isometric to one of $\mathbb{R}^m$, S^m, $\mathbb{RP}^m$, $\mathbb{H}^m$ (the m-dimensional hyperbolic space) – see [Ko2]. Riemannian manifolds with lower isometry group dimensions were extensively studied in the 1950's-70's and a substantial number of classification results were obtained. The proofs relied to a great extent on the properness of the $\text{Isom}(N)$-action on N. We refer the interested reader to [Ko2] and references therein for further details.

We now return to studying hyperbolic manifolds. Let $n := \dim_{\mathbb{C}} M$. First we will obtain an upper bound for $d(M)$; specifically, we will show that $d(M) \le n^2 + 2n$ (see [Ka], [Ko1]). For $p \in M$ define its *orbit* as $O(p) := \{gp : g \in G(M)\}$ and its *isotropy subgroup* as $I_p := \{g \in G(M) : gp = p\}$. The properness of the $G(M)$-action on M implies that $O(p)$ is a closed submanifold in M and that I_p is compact in $G(M)$ (see [Bi], [DK] for surveys on proper actions). Let $L_p := \{dg_p : g \in I_p\}$ be the *linear isotropy subgroup* where dg_p denotes the differential of a map $g \in I_p$ at p. Clearly, L_p is a subgroup of $GL(\mathbb{C}, T_p(M))$, where $T_p(M)$ is the tangent space to M at p. Consider the *isotropy representation* of I_p:

$$\alpha_p : I_p \to L_p, \quad g \mapsto dg_p. \tag{1.2}$$

The results of [Bo] (see also [MZ]) imply that α_p is faithful (this is also a consequence of the classical uniqueness theorem due to H. Cartan – see [CaH]). Since α_p is in addition continuous, it further follows that L_p is a compact subgroup of $GL(\mathbb{C}, T_p(M))$ isomorphic to I_p as a Lie group. In particular, there exist coordinates in $T_p(M)$ in which L_p becomes a subgroup of the unitary group U_n. Hence $\dim L_p \le n^2$. Therefore, we have

$$d(M) = \dim I_p + \dim O(p) \le n^2 + 2n. \tag{1.3}$$

We will first of all explore the largest possible automorphism group dimension $n^2 + 2n$. Let $\mathbb{B}^n$ denote the unit ball in $\mathbb{C}^n$ (we usually write Δ for $\mathbb{B}^1$). The group $\mathrm{Aut}(\mathbb{B}^n)$ consists of all maps of the form

$$z \mapsto \frac{Az + b}{cz + d}, \tag{1.4}$$

where

$$\begin{pmatrix} A & b \\ c & d \end{pmatrix} \in SU_{n,1}$$

(see e.g. [Ru]). It then follows that $\mathrm{Aut}(\mathbb{B}^n)$ is isomorphic to $PSU_{n,1} := SU_{n,1}/Z$, where Z is the center of $SU_{n,1}$, and therefore $d(\mathbb{B}^n) = n^2 + 2n$. The following converse implication is well-known.

Theorem 1.1. ([Ka], [Ko1]) *Let M be a connected hyperbolic manifold of dimension n. Suppose that $d(M) = n^2 + 2n$. Then M is holomorphically equivalent to $\mathbb{B}^n$.*

Before proving Theorem 1.1 we will give a useful definition. If $\mathcal{M}$ is any complex manifold, p a point in $\mathcal{M}$ and L a subgroup in $GL(\mathbb{C}, T_p(\mathcal{M}))$, then we say that L *acts transitively on real directions* in $T_p(\mathcal{M})$, if for any two non-zero vectors $v_1, v_2 \in T_p(\mathcal{M})$ there exists $g \in L$ such that $gv_1 = \lambda v_2$ for some $\lambda \in \mathbb{R}^*$. Similarly, we say that L *acts transitively on complex directions* in $T_p(\mathcal{M})$, if in the above definition λ is assumed to be complex.

Proof of Theorem 1.1: Since $d(M) = n^2 + 2n$, for every $p \in M$ we have $\dim L_p = n^2$, that is, in some coordinates in $T_p(M)$ the group L_p coincides with U_n. In particular, L_p acts transitively on real directions in $T_p(M)$. Further, since $d(M) > 0$, the manifold M is non-compact (otherwise M would contain an entire curve – see [Ko1]). By the result of [GK], every n-dimensional connected non-compact complex manifold $\mathcal{M}$ on which a compact group $K \subset \mathrm{Aut}(\mathcal{M})$ acts with a fixed point q in such a way that the action of the group $\alpha_q(K)$[1] is transitive on real directions in $T_q(\mathcal{M})$, is holomorphically equivalent to either $\mathbb{B}^n$ or $\mathbb{C}^n$. Since M is hyperbolic, we obtain that M is equivalent to $\mathbb{B}^n$, as required. ∎

Remark 1.2. The above proof is different from those given in [Ka], [Ko1]. The general result of [GK] that we have used, along with its analogue for compact manifolds obtained in [BDK], will be a standard tool throughout the book (see Theorems 1.7 and 1.11 in Sect. 1.4). Note that it is shown in [BDK] that if the dimension of the manifold is bigger than 1, it is sufficient to assume in Theorem 1.7 (cf. Theorem 1.11) that $\alpha_p(K)$ acts transitively on complex directions in $T_p(\mathcal{M})$ (see Remark 1.9). We also observe that, instead of using Theorem 1.7, in the proof of Theorem 1.1 we could refer to the classification of isotropic Riemannian manifolds in the way it was done in [Ak].

From now on we assume that $d(M) < n^2 + 2n$. Our goal is to find an *explicit* classification of hyperbolic manifolds with sufficiently large (certainly positive) values of $d(M)$, and we will now derive a lower bound for $d(M)$ beyond which an explicit classification for every n seems impossible. Since all Riemann surfaces with positive-dimensional automorphism group are well-known (see e.g. [FK]), we assume that $n \geq 2$. The lower bound can be obtained by considering *Reinhardt domains* in $\mathbb{C}^n$, that is, domains invariant under the rotations

$$z_j \mapsto e^{i\psi_j} z_j, \quad \psi_j \in \mathbb{R}, \quad j = 1, \ldots, n.$$

Most Reinhardt domains have no automorphisms other than these rotations and hence have an n-dimensional automorphism group. In particular, if D is a generic hyperbolic Reinhardt domain in $\mathbb{C}^2$, then $d(D) = 2 = n^2 - 2$. Such Reinhardt domains in $\mathbb{C}^2$ cannot be explicitly described, thus there exists no explicit classification of hyperbolic manifolds with $d(M) = n^2 - 2$ for $n = 2$. We will now make this argument more precise and give an example of a family of pairwise holomorphically non-equivalent smoothly bounded Reinhardt domains in $\mathbb{C}^2$ with automorphism group of dimension 2 (see [I2]). This family is parametrized by sets in $\mathbb{R}^2$ that satisfy only very mild conditions. In particular, no explicit formulas describe the domains in the family.

Choose a set $Q \subset \mathbb{R}^2_+ := \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ in such a way that the associated set in $\mathbb{C}^2$

$$D_Q := \big\{(z_1, z_2) \in \mathbb{C}^2 : (|z_1|, |z_2|) \in Q\big\}$$

is a smoothly bounded Reinhardt domain and contains the origin. By [Su], two bounded Reinhardt domains containing the origin are holomorphically equivalent if and only if one is obtained from the other by means of a dilation

[1] The isotropy representation is defined and continuous for any – not necessarily hyperbolic – complex manifold, and we will sometimes use this term, as well as the notation α_p, in this broader context. Here $\alpha_q(K) := \{dg_q : g \in K\}$.

and permutation of coordinates. In [FIK] all smoothly bounded Reinhardt domains with non-compact automorphism group were listed. All these domains contain the origin, and it is not difficult to choose Q so that D_Q is not holomorphically equivalent to any of the domains from [FIK] thus ensuring that $\text{Aut}(D_Q)$ is compact. It then follows from the explicit description of the automorphism groups of bounded Reinhardt domains (see [Kru], [Sh]) that $G(D_Q)$ is isomorphic to either U_2 or $U_1 \times U_1$. Theorem 1.9 of [GIK] now yields that there does not exist a hyperbolic Reinhardt domain in $\mathbb{C}^2$ containing the origin for which the automorphism group is compact and four-dimensional. Hence $G(D_Q)$ is in fact isomorphic to $U_1 \times U_1$ and thus $d(D_Q) = 2$. The freedom in choosing a set $Q \subset \mathbb{R}^2_+$ that satisfies the above requirements is very substantial, and by varying Q one can produce a family of pairwise non-equivalent Reinhardt domains D_Q that cannot be described by explicit formulas.

While it is possible that there is some classification for $n \geq 3$, $d(M) = n^2 - 2$, as well as for particular pairs n, $d(M)$ with $d(M) < n^2 - 2$ (see e.g. [GIK] for a study of Reinhardt domains from the point of view of automorphism group dimensions), we will be only interested in situations when an explicit classification exists for *all* values of n. In this book we give a complete explicit classification of connected hyperbolic n-dimensional manifolds with $d(M)$ within the following range:

$$n^2 - 1 \leq d(M) < n^2 + 2n. \tag{1.5}$$

1.3 A Lacuna in Automorphism Group Dimensions

In this section we will show that most values at the top of range (1.5) do not in fact realize.

Theorem 1.3. ([IKra2]) *There does not exist a connected hyperbolic manifold M of dimension $n \geq 2$ with $n^2 + 3 \leq d(M) < n^2 + 2n$.*

Proof: Suppose that M is a connected hyperbolic manifold M of dimension $n \geq 2$ with $n^2 + 3 \leq d(M) < n^2 + 2n$. Fix a point $p \in M$. We have

$$d(M) = \dim I_p + \dim O(p) \leq \dim I_p + 2n,$$

which yields the estimate

$$\dim I_p \geq n^2 - 2n + 3.$$

Recall that the group I_p is isomorphic to the linear isotropy subgroup $L_p \subset U_n \subset GL_n(\mathbb{C}) \simeq GL(\mathbb{C}, T_p(M))$ by means of the isotropy representation α_p. We now need the following lemma.

Lemma 1.4. ([IKra2]) *Let $G \subset U_n$ be a connected closed subgroup, $n \geq 2$. Suppose that $\dim G \geq n^2 - 2n + 3$. Then either $G = U_n$, or $G = SU_n$, or $n = 4$ and G is conjugate in U_4 to $e^{i\mathbb{R}} Sp_2$, where Sp_2 denotes the compact real form of $Sp_4(\mathbb{C})$.*[2]

[2] Here we follow the notation introduced in [VO].

Proof of Lemma 1.4: Since G is compact, its natural representation on $\mathbb{C}^n$ is completely reducible, that is, $\mathbb{C}^n$ splits into the sum of G-invariant pairwise orthogonal complex subspaces:

$$\mathbb{C}^n = V_1 \oplus \ldots \oplus V_k,$$

such that for every j the restriction G_j of G to V_j defines an irreducible representation of G. Let $n_j := \dim_{\mathbb{C}} V_j$ (we have $n_1 + \ldots + n_k = n$), and U_{n_j} be the group of unitary transformations of V_j. $G_j \subset U_{n_j}$ and therefore $\dim G_j \le n_j^2$ for every j. Hence we have

$$n^2 - 2n + 3 \le \dim G \le n_1^2 + \ldots + n_k^2.$$

It follows from the above inequalities that $k = 1$, that is, G acts irreducibly on $\mathbb{C}^n$.

Let $\mathfrak{g} \subset \mathfrak{u}_n \subset \mathfrak{gl}_n$ be the Lie algebra of G and $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} + i\mathfrak{g} \subset \mathfrak{gl}_n$ its complexification (here and everywhere below $\mathfrak{gl}_n$ denotes the Lie algebra of all matrices of size n with complex elements). Since G is connected, $\mathfrak{g}^{\mathbb{C}}$ acts irreducibly on $\mathbb{C}^n$ and, by a theorem of E. Cartan (see e.g. [GG]), $\mathfrak{g}^{\mathbb{C}}$ either is semisimple or is the direct sum of a semisimple ideal $\mathfrak{h}$ and the center of $\mathfrak{gl}_n$ (which is isomorphic to $\mathbb{C}$). Clearly, in the latter case the action of $\mathfrak{h}$ on $\mathbb{C}^n$ is irreducible.

Suppose first that $\mathfrak{g}^{\mathbb{C}}$ is semisimple. Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_m$ be the decomposition of $\mathfrak{g}^{\mathbb{C}}$ into the direct sum of simple ideals. Then the natural irreducible n-dimensional representation of $\mathfrak{g}^{\mathbb{C}}$ (given by the embedding of $\mathfrak{g}^{\mathbb{C}}$ in $\mathfrak{gl}_n$) is the tensor product of some irreducible faithful representations of the ideals $\mathfrak{g}_j$ (see e.g. [GG]). Let l_j be the dimension of the corresponding representation of $\mathfrak{g}_j$, $j = 1, \ldots, m$. Then $l_j \ge 2$, $\dim_{\mathbb{C}} \mathfrak{g}_j \le l_j^2 - 1$, and $n = l_1 \cdot \ldots \cdot l_m$. The following observation is straightforward.

Claim: *If $n = l_1 \cdot \ldots \cdot l_m$, $m \ge 2$, $l_j \ge 2$ for $j = 1, \ldots, m$, then $\sum_{j=1}^m l_j^2 \le n^2 - 2n$.*

It follows from the claim that $m = 1$, i.e. $\mathfrak{g}^{\mathbb{C}}$ is simple. The minimal dimensions of irreducible faithful representations of complex simple Lie algebras $\mathfrak{s}$ are well-known (see e.g. [VO]). In the table below V denotes the representation space of minimal dimension.

$\mathfrak{s}$	$\dim V$	$\dim \mathfrak{s}$
$\mathfrak{sl}_k$ $k \ge 2$	k	$k^2 - 1$
$\mathfrak{o}_k$ $k \ge 7$	k	$k(k-1)/2$
$\mathfrak{sp}_{2k}$ $k \ge 2$	$2k$	$2k^2 + k$
$\mathfrak{e}_6$	27	78
$\mathfrak{e}_7$	56	133
$\mathfrak{e}_8$	248	248
$\mathfrak{f}_4$	26	52
$\mathfrak{g}_2$	7	14

Since $\dim_{\mathbb{C}}\mathfrak{g}^{\mathbb{C}} \geq n^2 - 2n + 3$, it follows that $\mathfrak{g}^{\mathbb{C}} \simeq \mathfrak{sl}_n$, which implies that $\mathfrak{g} \simeq \mathfrak{su}_n$, and therefore $G = SU_n$.

Suppose now that $\mathfrak{g}^{\mathbb{C}} = \mathbb{C} \oplus \mathfrak{h}$, where $\mathfrak{h}$ is a semisimple ideal in $\mathfrak{g}^{\mathbb{C}}$. Then, repeating the above argument for $\mathfrak{h}$ in place of $\mathfrak{g}^{\mathbb{C}}$ and taking into account that $\dim_{\mathbb{C}}\mathfrak{h} \geq n^2 - 2n + 2$, we conclude that $\mathfrak{h} \simeq \mathfrak{sl}_n$ for $n \neq 4$. Therefore, for $n \neq 4$, we have $\mathfrak{g} = \mathfrak{gl}_n$ and hence $G = U_n$. For $n = 4$ we have either $\mathfrak{h} \simeq \mathfrak{sl}_4$ or $\mathfrak{h} \simeq \mathfrak{sp}_4$. We thus find that either $G = U_4$ or $\mathfrak{g}^{\mathbb{C}} \simeq \mathbb{C} \oplus \mathfrak{sp}_4$, respectively. Further, we observe that every irreducible 4-dimensional representation of $\mathfrak{sp}_4$ is equivalent to its defining representation (which is the first fundamental representation of $\mathfrak{sp}_4$). This implies that G is conjugate in $GL_4(\mathbb{C})$ to $e^{i\mathbb{R}}Sp_2$. Since $G \subset U_4$, it is straightforward to show that the conjugating element can be chosen to belong to U_4 (in fact, one can prove that every element $g \in GL_4(\mathbb{C})$ such that $gSp_2g^{-1} \subset U_4$, has the form $g = \lambda U$, where $\lambda > 0$ and $U \in U_4$).

The proof of Lemma 1.4 is complete. ■

Theorem 1.3 immediately follows from Lemma 1.4. Indeed, the lemma implies that L_p acts transitively on real directions in $T_p(M)$ (note that the standard action of Sp_2 on the 7-dimensional sphere in $\mathbb{C}^4$ is transitive) and, arguing as in the proof of Theorem 1.1 (see also Remark 1.2), we obtain that M is holomorphically equivalent to $\mathbb{B}^n$. Clearly, this is impossible, and the proof of the theorem is complete. ■

In the forthcoming chapters we will classify connected hyperbolic n-dimensional manifolds M for which $n^2 - 1 \leq d(M) \leq n^2 + 2$. We will separately consider two cases: that of homogeneous manifolds and that of non-homogeneous ones. Here a connected manifold M is called Aut(M)-*homogeneous* or simply *homogeneous* if the group Aut(M) acts *transitively* on M, where we say in general that a group G acts transitively on a manifold N if for every $p, q \in N$ there exists an element $g \in G$ such that $gp = q$ (observe that for a connected N this implies that the group G^0 acts on N transitively as well). The homogeneous case is considered in the next chapter, and the more involved non-homogeneous case occupies Chaps. 3–5.

Before proceeding with our solution to the classification problem, we survey the main tools used throughout the book.

1.4 Main Tools

To deal with the homogeneous case we use the theory of *Siegel domains of the second kind* in $\mathbb{C}^n$, i.e. domains of the form

$$\left\{(z,w) \in \mathbb{C}^{n-k} \times \mathbb{C}^k : \operatorname{Im} w - F(z,z) \in C\right\}, \tag{1.6}$$

where $1 \leq k \leq n$, C is an open convex cone in $\mathbb{R}^k$ not containing an entire line, and $F = (F_1, \ldots, F_k)$ is a $\mathbb{C}^k$-valued Hermitian form on $\mathbb{C}^{n-k} \times \mathbb{C}^{n-k}$

such that $F(z,z) \in \overline{C} \setminus \{0\}$ for all non-zero $z \in \mathbb{C}^{n-k}$. Observe that for $k = 1$ all such domains are equivalent to $\mathbb{B}^n$. Siegel domains of the second kind were introduced by Pyatetskii-Shapiro (see [P-S]) with the purpose of generalizing the classical symmetric domains (all symmetric domains were determined by E. Cartan in [CaE2]), and therefore sometimes are called the *Pyatetskii-Shapiro domains* in the literature. For brevity we call them *Siegel domains*. The monographs [P-S] and [Sa] are a good source of information on such domains.

Siegel domains are unbounded by definition, but in fact every such domain has a bounded realization and hence is hyperbolic. We are interested in these domains since it turns out that every homogeneous hyperbolic manifold is holomorphically equivalent to a Siegel domain (note, however, that not every Siegel domain is homogeneous). This result is due to Nakajima (see [N]) and solves a problem posed by Kobayashi (see [Ko1]). The techniques used to obtain it go back to the fundamental theorem of Vinberg, Gindikin and Pyatetskii-Shapiro for homogeneous bounded domains (see [P-S]).

To determine all homogeneous Siegel domains D with $n^2 - 1 \le d(D) \le n^2 + 2$ we utilize the well-known description of the Lie algebra $\mathfrak{g}(D)$ of the group $\mathrm{Aut}(D)$. We will write it as a Lie algebra of certain holomorphic vector fields on D.

In general, if a Lie group G acts effectively on a complex manifold M by holomorphic transformations, then the Lie algebra $\mathfrak{g}$ of G is isomorphic to the Lie algebra of *G-vector fields* on M. A holomorphic vector field X on M is called a G-vector field if there exists $a \in \mathfrak{g}$ such that for all $p \in M$ we have

$$X(p) = \frac{d}{dt}\Big[\exp(ta)p\Big]\Big|_{t=0}.$$

By [KMO], for a Siegel domain D the algebra $\mathfrak{g}(D)$ is a graded Lie algebra:

$$\mathfrak{g}(D) = \mathfrak{g}_{-1}(D) \oplus \mathfrak{g}_{-1/2}(D) \oplus \mathfrak{g}_0(D) \oplus \mathfrak{g}_{1/2}(D) \oplus \mathfrak{g}_1(D), \qquad (1.7)$$

where $\mathfrak{g}_{-1}(D)$ consists of all vector fields of the form

$$b\frac{\partial}{\partial w}, \quad b \in \mathbb{R}^k, \qquad (1.8)$$

$\mathfrak{g}_{-1/2}(D)$ consists of all vector fields of the form

$$c\frac{\partial}{\partial z} + 2iF(z,c)\frac{\partial}{\partial w}, \quad c \in \mathbb{C}^{n-k}, \qquad (1.9)$$

and $\mathfrak{g}_0(D)$ is the Lie algebra of all vector fields of the form

$$Az\frac{\partial}{\partial z} + Bw\frac{\partial}{\partial w}, \qquad (1.10)$$

where $A \in \mathfrak{gl}_{n-k}$, B belongs to the Lie algebra $\mathfrak{g}_{\mathrm{lin}}(C)$ of the group $G_{\mathrm{lin}}(C)$ of linear automorphisms of the cone C, and the following holds:

$$F(Az, z) + F(z, Az) = BF(z, z), \tag{1.11}$$

for all $z \in \mathbb{C}^{n-k}$. Moreover, $\dim \mathfrak{g}_j(D) \le \dim \mathfrak{g}_{-j}(D)$ for $j = 1/2, 1$. Furthermore, the components $\mathfrak{g}_{1/2}(D)$ and $\mathfrak{g}_1(D)$ also admit explicit descriptions. These descriptions are quite involved and to avoid giving bulky formulas here we refer the reader to [Sa], pp. 213–219. Using the above structure of $\mathfrak{g}(D)$ together with the constraint $n^2 - 1 \le d(D) \le n^2 + 2$ we show that k cannot be large (in most cases $k \le 2$). This allows us to explicitly find all possible cones C and eventually determine all homogeneous Siegel domains D with $n^2 - 1 \le d(D) \le n^2 + 2$ in Theorem 2.2. We remark here that this technique also works for some values of $d(D)$ below $n^2 - 1$.

We will now turn to the non-homogeneous case. It will be shown in Proposition 2.1 that non-homogeneous manifolds can only occur for $d(M) = n^2 - 1$ and $d(M) = n^2$. To obtain a classification in these cases we first classify the orbits of the $G(M)$-action on M and then join them together to form hyperbolic manifolds. The classification of orbits is done, in part, by utilizing the methods of *CR-geometry*, and we will now give a brief introduction to it. Comprehensive surveys on CR-geometry can be found in [Tu], [Ch].

A *CR-structure* on a smooth connected manifold N of dimension m is a smooth distribution of subspaces in the tangent spaces $T_p^c(N) \subset T_p(N)$, $p \in N$ (a subbundle $T^c(N)$ of the tangent bundle $T(N)$), with operators of complex structure $J_p : T_p^c(N) \to T_p^c(N)$, $J_p^2 \equiv -\mathrm{id}$, that depend smoothly on p. A manifold N equipped with a CR-structure is called a *CR-manifold*. The number $CR\dim N := \dim_{\mathbb{C}} T_p^c(N)$ does not depend on p and is called the *CR-dimension* of N. The number $CR\,\mathrm{codim}\, N := m - 2CR\dim N$ is called the *CR-codimension* of N. Clearly, if N is a complex manifold (hence m is even), it is also a CR-manifold of CR-dimension $m/2$ and CR-codimension 0. At the other extreme, if for a CR-manifold N we have $CR\dim N = 0$, then N is called *totally real*. CR-structures naturally arise on real submanifolds of complex manifolds. Indeed, if N is a real submanifold of a complex manifold M, then one can define the distribution $T_p^c(N)$ as follows:

$$T_p^c(N) := T_p(N) \cap \mathcal{J}_p T_p(N),$$

where $\mathcal{J}_p$ is the operator of complex structure in $T_p(M)$. On each $T_p^c(N)$ the operator J_p is then defined as the restriction of $\mathcal{J}_p$ to $T_p^c(N)$. Then $\{T_p^c(N), J_p\}_{p \in N}$ is a CR-structure on N, provided $\dim_{\mathbb{C}} T_p^c(N)$ is constant. This is always the case, for example, if N is a real hypersurface in M (in which case $CR\,\mathrm{codim}\, N = 1$). We say that such a CR-structure is *induced by* M, or that N is a *CR-submanifold of* M.

A smooth map between two CR-manifolds $f : N_1 \to N_2$ is called a *CR-map*, if for every $p \in N_1$: (i) df_p maps $T_p^c(N_1)$ to $T_{f(p)}^c(N_2)$, and (ii) df_p is complex-linear on $T_p^c(N_1)$. If for $j = 1, 2$ we have $N_j \subset M_j$ where M_j is a complex manifold and the CR-structure of N_j is induced by M_j, then for every holomorphic map $F : M_1 \to M_2$ such that $F(N_1) \subset N_2$, the restriction

$F|_{N_1}$ is a CR-map from N_1 into N_2. Two CR-manifolds N_1, N_2 are called *CR-equivalent*, if there is a diffeomorphism from N_1 onto N_2 which is a CR-map. Such a diffeomorphism is called a *CR-isomorphism*, or *CR-equivalence*. A CR-isomorphism of a CR-manifold onto itself is called a *CR-automorphism*. We denote by $\mathrm{Aut}_{CR}(N)$ the group of all CR-automorphisms of a CR-manifold N and equip it with the compact-open topology. The manifold N is called *CR-homogeneous* if the group $\mathrm{Aut}_{CR}(N)$ acts transitively on N.

Let N be a CR-manifold. For every $p \in N$ consider the complexification $T_p^c(N) \otimes_{\mathbb{R}} \mathbb{C}$. Clearly, it can be represented as the direct sum

$$T_p^c(N) \otimes_{\mathbb{R}} \mathbb{C} = T_p^{(1,0)}(N) \oplus T_p^{(0,1)}(N),$$

where

$$\begin{aligned} T_p^{(1,0)}(N) &:= \{X - iJ_pX : X \in T_p^c(N)\}, \\ T_p^{(0,1)}(N) &:= \{X + iJ_pX : X \in T_p^c(N)\}. \end{aligned}$$

The CR-structure on N is called *integrable* if for any local sections Z, Z' of the bundle $T^{(1,0)}(N)$, the vector field $[Z, Z']$ is also a local section of $T^{(1,0)}(N)$. It is not difficult to see that if N is a CR-submanifold of a complex manifold M then the CR-structure on N is integrable. We only consider integrable CR-structures.

An important characteristic of a CR-structure called the *Levi form* comes from taking commutators of local sections of $T^{(1,0)}(N)$ and $T^{(0,1)}(N)$. Let $p \in N$, $z, z' \in T_p^{(1,0)}(N)$, and Z, Z' be local sections of $T^{(1,0)}(N)$ near p such that $Z(p) = z$, $Z'(p) = z'$. The Levi form of N at p is the Hermitian form on $T_p^{(1,0)}(N)$ with values in $(T_p(N)/T_p^c(N)) \otimes_{\mathbb{R}} \mathbb{C}$ given by

$$\mathcal{L}_N(p)(z, z') := i[Z, \overline{Z'}](p) (\operatorname{mod} T_p^c(N) \otimes_{\mathbb{R}} \mathbb{C}).$$

The Levi form is defined uniquely up to the choice of coordinates in $(T_p(N)/T_p^c(N)) \otimes_{\mathbb{R}} \mathbb{C}$, and, for fixed z and z', its value does not depend on the choice of Z and Z'. The Levi-form is invariant under CR-isomorphisms up to the choice of coordinates in $(T_p(N)/T_p^c(N)) \otimes_{\mathbb{R}} \mathbb{C}$.

If for a CR-manifold N we have $\mathcal{L}_N \equiv 0$, then N is called *Levi-flat*. For a Levi-flat CR-manifold N the distribution $\{T_p^c(N)\}_{p \in N}$ is involutive, that is, if X and Y are local sections of $T^c(N)$, then so is $[X, Y]$. The maximal connected integral manifolds of this distribution form a foliation of N (see [St]). They are (not necessarily closed) almost complex submanifolds of N and are called the *leaves of the foliation*. For $p \in N$ we denote by N_p the leaf passing through p. Due to the integrability of the CR-structure on N, the almost complex structure on every leaf is also integrable, and by a theorem of Newlander and Nirenberg (see [NN]), the leaves are in fact complex manifolds. It can now be shown (see [Ch]) that $\{N_p\}_{p \in N}$ is a *complex foliation* of N, that is, for every $p \in N$ there exists a neighborhood $\mathcal{V}$ of p and a diffeomorphism φ from $\mathcal{V}$ onto $\mathbb{B}^k \times U$, where $k := CR \dim N$ and U is the unit ball in $\mathbb{R}^{m-2k}$, such that for every $q \in \mathcal{V}$ the map φ maps $N_q \cap \mathcal{V}$ biholomorphically onto $\mathbb{B}^k \times \{t\}$ for some $t \in U$ (it clearly follows that φ is a CR-isomorphism).

Suppose now that $CR \operatorname{codim} N = 1$. Then the dimension m is odd, and we assume that $m \geq 3$ (hence $CR \dim N \geq 1$). If for a point $p \in N$ one can choose coordinates in $(T_p(N)/T_p^c(N)) \otimes_{\mathbb{R}} \mathbb{C}$ in such a way that $\mathcal{L}_N(p)$ is a positive-definite Hermitian form, the manifold M is called *strongly pseudoconvex at* p. The manifold N is called *strongly pseudoconvex* if N is strongly pseudoconvex at its every point. For example, if $m \geq 3$ is odd, the unit sphere S^m is a strongly pseudoconvex hypersurface in $\mathbb{C}^{(m+1)/2}$ with the induced CR-structure. For a strongly pseudoconvex CR-manifold N the group $\operatorname{Aut}_{CR}(N)$ is known to be a Lie group (see [Ta], [CM], [BS2], [Sch]). It then follows that a strongly pseudoconvex CR-homogeneous CR-manifold is real-analytic.

We will be interested, in particular, in so-called spherical CR-manifolds. A CR-manifold N of dimension $m \geq 3$ and CR-codimension 1 is called *spherical at a point* p, if there exists a neighborhood of p which is CR-equivalent to an open subset of S^m; otherwise N is called *non-spherical at* p. The manifold N is called *spherical* if it is spherical at its every point; otherwise N is called *non-spherical*.

We will now give sufficient conditions for the sphericity of a closed real-analytic strongly pseudoconvex hypersurface N in a complex manifold M of dimension $n \geq 2$ with the induced CR-structure (here $m = 2n - 1$). Fix $p \in N$. In some holomorphic coordinates $z = (z_1, \dots, z_{n-1})$, $w = u + iv$ in a neighborhood of p in M (in which p is the origin), the hypersurface N can be written in the *Chern-Moser normal form* (see [CM]), that is, given by an equation[3]

$$v = |z|^2 + F(z, \overline{z}, u),$$

with

$$F(z, \overline{z}, u) = \sum_{k, \bar{l} \geq 2} F_{k\bar{l}}(z, \overline{z}, u),$$

where $F_{k\bar{l}}(z, \overline{z}, u)$ are polynomials of degree k in z and $\bar{l}$ in $\overline{z}$ whose coefficients are analytic functions of u such that the following conditions hold:

$$\begin{aligned} \operatorname{tr} F_{2\overline{2}} &\equiv 0, \\ \operatorname{tr}^2 F_{2\overline{3}} &\equiv 0, \\ \operatorname{tr}^3 F_{3\overline{3}} &\equiv 0; \end{aligned} \tag{1.12}$$

here the operator tr is defined as

$$\operatorname{tr} := \sum_{\alpha=1}^{n-1} \frac{\partial^2}{\partial z_\alpha \partial \overline{z}_\alpha}.$$

Coordinates in which N is represented in the Chern-Moser normal form near p are called *normal coordinates at* p. For $n \geq 3$ the point p is called *umbilic*,

[3] Here and everywhere below $|\cdot|$ denotes the standard norm in $\mathbb{C}^k$ for any k.

if $F_{2\overline{2}} \equiv 0$ for some choice of normal coordinates at p. The point p is called umbilic for $n = 2$ if in some normal coordinates at p we have $F_{2\overline{4}} \equiv 0$ (note that it follows from conditions (1.12) that $F_{2\overline{2}} \equiv F_{2\overline{3}} \equiv F_{3\overline{3}} \equiv 0$ if $n = 2$). The definition of umbilic point is in fact independent of the choice of normal coordinates. If N is spherical at p then in any normal coordinates at p we have $F \equiv 0$, hence every point in a neighborhood of p in N is umbilic. Conversely, it is well-known (see e.g. [Tu]) that N is spherical at p if every point in a neighborhood of p in N is umbilic.

Let $\mathrm{Aut}_p(N)$ denote the group of all local CR-automorphisms of N defined near p and preserving it. This group is often called the *stability group of N at p*. Fix normal coordinates (z, w) at p. Since N is real-analytic, every element φ of $\mathrm{Aut}_p(N)$ extends to a biholomorphic map defined in a neighborhood of p in M (see [BJT]), and therefore in the coordinates (z, w) can be written as

$$\begin{aligned} z &\mapsto f_\varphi(z, w), \\ w &\mapsto g_\varphi(z, w), \end{aligned}$$

where f_φ and g_φ are holomorphic near p. We equip $\mathrm{Aut}_p(N)$ with the topology of uniform convergence of the partial derivatives of all orders of the component functions on a neighborhood of p in N. The group $\mathrm{Aut}_p(N)$ with this topology is a topological group.

Assume first that N is spherical at p. Due to the Poincaré-Alexander theorem (see [Alex]), every CR-isomorphism between open subsets of S^{2n-1} extends to an element of $\mathrm{Aut}(\mathbb{B}^n)$. It then follows from formula (1.4) that $\mathrm{Aut}_p(N)$ is a Lie group of dimension $n^2 + 1$.

Assume now that N is non-spherical at p. It it shown in [CM] that every element $\varphi = (f_\varphi, g_\varphi)$ of $\mathrm{Aut}_p(N)$ is uniquely determined by a set of parameters $(U_\varphi, a_\varphi, \lambda_\varphi, r_\varphi)$, where $U_\varphi \in U_{n-1}$, $a_\varphi \in \mathbb{C}^{n-1}$, $\lambda_\varphi > 0$, $r_\varphi \in \mathbb{R}$. These parameters are found from the following relations:

$$\lambda_\varphi U_\varphi = \frac{\partial f_\varphi}{\partial z}(0), \, \lambda_\varphi U_\varphi a_\varphi = \frac{\partial f_\varphi}{\partial w}(0),$$

$$\lambda_\varphi^2 = \frac{\partial g_\varphi}{\partial w}(0), \qquad \lambda_\varphi^2 r_\varphi = \mathrm{Re}\, \frac{\partial^2 g_\varphi}{\partial^2 w}(0).$$

By the results of [Be], [Lo], [BV] (see also [CM]), the non-sphericity of N at p implies that for every element $\varphi = (f_\varphi, g_\varphi)$ of $\mathrm{Aut}_p(N)$ one has $\lambda_\varphi = 1$ and the parameters a_φ and r_φ are uniquely determined by the matrix U_φ. Moreover, the map

$$\mathfrak{P} : \mathrm{Aut}_p(N) \to GL_{n-1}(\mathbb{C}), \qquad \mathfrak{P} : \varphi \mapsto U_\varphi$$

is a topological group isomorphism between $\mathrm{Aut}_p(N)$ and $G_p(N) := \mathfrak{P}(\mathrm{Aut}_p(N))$ with $G_p(N)$ being a real-algebraic subgroup of $GL_{n-1}(\mathbb{C})$. The subgroup $G_p(N)$ is closed in $GL_{n-1}(\mathbb{C})$, and we can pull back its Lie group structure to $\mathrm{Aut}_p(N)$ by means of $\mathfrak{P}$. Let $d_p(N)$ denote the dimension of $\mathrm{Aut}_p(N)$ (equal to that of $G_p(N)$). Since $G_p(N)$ is a closed subgroup of U_{n-1}, we have $d_p(N) \leq (n-1)^2$, and if $d_p(N) = (n-1)^2$, then $G_p(N) = U_{n-1}$.

Furthermore, the non-sphericity of N at p yields that the group $\mathrm{Aut}_p(N)$ is *linearizable*, that is, in some normal coordinates at p every $\varphi \in \mathrm{Aut}_p(N)$ can be written in the form

$$\begin{aligned} z &\mapsto Uz, \\ w &\mapsto w \end{aligned} \tag{1.13}$$

for some $U \in U_{n-1}$ (see [KL]). If all elements of $\mathrm{Aut}_p(N)$ in some coordinates at p have the form (1.13), we say that $\mathrm{Aut}_p(N)$ is *linear* in these coordinates.

Suppose now that $d_p(N) = (n-1)^2$. It then follows that, in normal coordinates at p in which $\mathrm{Aut}_p(N)$ is linear, the function F is invariant under all unitary transformations in the z-variables and therefore depends only on $|z|^2$ and u. Conditions (1.12) then imply that $F_{2\overline{2}} \equiv 0$, $F_{3\overline{3}} \equiv 0$. Thus, F has the form

$$F(z, \overline{z}, u) = \sum_{k=4}^{\infty} C_k(u)|z|^{2k}, \tag{1.14}$$

where $C_k(u)$ are real-valued analytic functions of u (and due to the non-sphericity of N at p for some k we have $C_k(u) \not\equiv 0$). Hence the following proposition holds (where for N spherical at p we set $d_p(N) := n^2 + 1$).

Proposition 1.5. *Let N be a closed real-analytic strongly pseudoconvex hypersurface in a complex manifold M of dimension $n \geq 2$ and p a point in N. Then $d_p(N) \geq (n-1)^2$ implies that p is umbilic. If, in addition, N is CR-homogeneous, then it is spherical.*

Furthermore, we will now show that the following is true (see [EzhI] for more results in this direction).

Proposition 1.6. ([EzhI])*Let N be a closed real-analytic strongly pseudoconvex hypersurface in a complex manifold M of dimension $n \geq 3$ and p a point in N. Then $d_p(N) \geq n^2 - 4n + 6$ implies that $d_p(N) \geq (n-1)^2$.*

Proof: We only need to consider the case when N is non-spherical at p. If we write N near p in normal coordinates at p in which $\mathrm{Aut}_p(N)$ is linear, then the function F is invariant under all transformations from the closed subgroup $G_p(N) \subset U_{n-1}$. We have $\dim G_p(N) \geq n^2 - 4n + 6 = (n-1)^2 - 2(n-1) + 3$, and Lemma 1.4 yields that either $G_p(N) = U_{n-1}$ or $G_p(N)^0 = SU_{n-1}$. In the first case we have $d_p(N) = (n-1)^2$. Assume now that $G_p(N)^0 = SU_{n-1}$. Then the function F is invariant under all transformations in the z-variables given by matrices from SU_{n-1} and therefore depends only on $|z|^2$ and u (recall that $n \geq 3$). Thus F has the form (1.14), which implies that in fact $G_p(N) = U_{n-1}$ contradicting our assumption. The proof is complete. ■

In this book CR-manifolds appear as the orbits of the action of the group $G(M)$ on a hyperbolic manifold M. Indeed, since $G(M)$ acts properly by holomorphic transformations, for every $p \in M$ its orbit $O(p)$ is a closed connected real-analytic CR-homogeneous CR-submanifold of M.

Assume first that $d(M) = n^2$ (this case is considered in Chap. 3). In the proof of Proposition 3.2 we show by a simple dimension-counting argument (that has its origin in the proof of Proposition 2.1) that for every $p \in M$ the following holds: (i) the codimension of $O(p)$ in M is either 1 or 2; (ii) L_p acts transitively on real directions in $T_p^c(O(p))$; (iii) if the codimension of $O(p)$ is 2, then $O(p)$ is a complex hypersurface in M. Statement (ii) implies that if the codimension of $O(p)$ is 1, then $O(p)$ is either strongly pseudoconvex or Levi-flat.

Suppose first that the codimension of $O(p)$ is 1 and $O(p)$ is strongly pseudoconvex. Then $\dim I_p = (n-1)^2$, which yields that $d_p(O(p)) \geq (n-1)^2$, and hence $O(p)$ is spherical by Proposition 1.5. Therefore, the universal cover $\tilde{O}(p)$ of $O(p)$ covers a homogeneous domain $D \subset S^{2n-1}$ (see [BS1]), and, since $d(M) \geq n^2$, we have $\dim \mathrm{Aut}_{CR}(D) \geq n^2$. All homogeneous domains in S^{2n-1} were found in [BS1], and it is not difficult to single out those for which $\dim \mathrm{Aut}_{CR}(D) \geq n^2$. The universal covers of such domains give a list of all possibilities for $\tilde{O}(p)$ which leads to a classification (up to CR-equivalence) of all strongly pseudoconvex orbits (see Proposition 3.3) together with actions of $G(M)$ on them (see Proposition 3.4).

Further, in order to classify codimension 1 Levi-flat orbits, as well as orbits of codimension 2, we need the following theorem.

Theorem 1.7. ([GK])*Let $\mathcal{M}$ be a connected non-compact complex manifold of dimension $k \geq 1$ and p a point in $\mathcal{M}$. Let $K \subset Aut(\mathcal{M})$ be a compact subgroup that fixes p and assume that the group $\alpha_p(K)$ acts transitively on real directions in $T_p(\mathcal{M})$. Then $\mathcal{M}$ is holomorphically equivalent to either $\mathbb{B}^k$ or $\mathbb{C}^k$.*

Recall that we used Theorem 1.7 in the proof of Theorem 1.1 (see also Remark 1.2). We will sketch the proof of Theorem 1.7, as well as that of its analogue for compact manifolds (Theorem 1.11), at the end of this section.

Assume now that $O(p)$ is a codimension 1 Levi-flat orbit. Then $O(p)$ is foliated by $(n-1)$-dimensional (not necessarily closed) connected complex submanifolds of M. Consider the leaf $O(p)_p$ of the foliation passing through p. Since M is hyperbolic and the Kobayashi pseudodistance does not increase under holomorphic maps (see (1.1)), the manifold $O(p)_p$ is hyperbolic as well. Further, the group I_p acts on $O(p)_p$ in such a way that the action of L_p on real directions in $T_p^c(O(p)) = T_p(O(p)_p)$ is transitive. Therefore, we have $d(O(p)_p) > 0$, which implies that $O(p)_p$ is non-compact (see [Ko1]). Now Theorem 1.7 yields that $O(p)_p$ is holomorphically equivalent to $\mathbb{B}^{n-1}$. Clearly, this assertion holds for every leaf of the foliation of $O(p)$. Using this fact, we show in Proposition 3.5 that $O(p)$ is CR-equivalent to either $\mathbb{B}^{n-1} \times S^1$ or $\mathbb{B}^{n-1} \times \mathbb{R}$ and determine the action of $G(M)$ on $O(p)$ (later we will see that the second possibility in fact does not realize). We emphasize that Proposition 3.5 is one of the main statements of Chap. 3 and that its proof is non-trivial. Generally, dealing with orbits that are not strongly pseudoconvex is the hardest part of our arguments throughout the book.

Next, if $O(p)$ is a complex hypersurface in M, the argument that we have just applied to $O(p)_p$ gives that $O(p)$ is holomorphically equivalent to $\mathbb{B}^{n-1}$. We also remark (see the proof of Proposition 3.2) that there are at most two orbits of codimension 2 in M. This follows from the existence of at least one orbit of codimension 1 and from the well-known description of the orbit space of proper group actions with codimension 1 orbits (see [AA]).

Having obtained the above classification of orbits, we study ways in which they can be joined together to form a hyperbolic manifold M with $d(M) = n^2$. For this purpose we develop an orbit gluing procedure that describes how orbits of codimension 1 can be glued into a hyperbolic manifold. This procedure is stated in detail in Sect. 3.4. Using the procedure we list all possibilities for the manifold M' which is obtained from M by removing all (at most two) orbits of codimension 2. Finally, if a codimension 2 orbit O is present in M, for every $p_0 \in O$ we introduce the I_{p_0}-invariant set

$$K_{p_0} := \{p \in M' : I_p \subset I_{p_0}\} \tag{1.15}$$

that approaches O at p_0 and use these sets (that turn out to be complex curves) to attach orbits of codimension 2 to the known possibilities for M' (see Sect. 3.4). As a result of our considerations, it turns out that in fact only one orbit of codimension 2 can occur. For any of the resulting hyperbolic manifolds M we have $d(M) \geq n^2$, and in order to obtain a complete classification for the case $d(M) = n^2$, it only remains to rule out manifolds with $d(M) > n^2$. This finalizes the proof of the main result of Chap. 3 (Theorem 3.1).

Assume now that $d(M) = n^2 - 1$ and suppose first that $n \geq 3$ (this case is considered in Chap. 4). To obtain an analogue of Proposition 3.2, we use a similar dimension-counting argument together with Theorem 1.7, as well as a description of all $(n-1)^2$-dimensional connected closed subgroups of U_n obtained in Lemma 2.1 of [IKru1]. We show in the proof of Proposition 4.2 that for every $p \in M$ the following holds: (i) the codimension of $O(p)$ in M is either 1 or 2; (ii) L_p acts transitively on real directions in $T_p^c(O(p))$; (iii) every orbit of codimension 1 is spherical; (iv) if the codimension of $O(p)$ is 2, then $O(p)$ is a complex hypersurface in M. In particular, in (iii) we prove that Levi-flat orbits cannot occur in this case. This is done by an argument that uses some elements of the proof of Proposition 3.5. Further, to see that every strongly pseudoconvex orbit is spherical, we utilize Propositions 1.5, 1.6.

Next, spherical orbits are described as in Proposition 3.3 (see Proposition 4.4) and Theorem 1.7 yields, as before, that every orbit of codimension 2 is holomorphically equivalent to $\mathbb{B}^{n-1}$. Observe, however, that in contrast with the case $d(M) = n^2$, for $d(M) = n^2 - 1$ every orbit in M may have codimension 2. Further, for manifolds with orbits of codimension 1 we argue as earlier, firstly, by utilizing our orbit gluing procedure from Sect. 3.4 and, secondly, by attaching codimension 2 orbits to the resulting manifolds M'. However, it turns out that for all manifolds obtained by this process one has $d(M) \geq n^2$. This contradiction shows that orbits of codimension 1 in fact do not occur for $d(M) = n^2 - 1$, $n \geq 3$ (see Theorem 4.3). Once we know that every orbit in M has codimension 2, it is possible to show that M is holomorphically equivalent to a product $\mathbb{B}^{n-1} \times S$, where S is a hyperbolic Riemann surface with $d(S) = 0$ (see Sect. 4.4). This is the main result of Chap. 4 (see Theorem 4.1).

We will now discuss connected hyperbolic manifolds for which $n = 2$ and $d(M) = 3$ (see Chap. 5). For brevity we call such manifolds (2,3)-*manifolds*. For a (2,3)-manifold M we show in the proof of Proposition 4.2 that for every $p \in M$ the following holds: (i) the codimension of $O(p)$ in M is either 1 or 2; (ii) every codimension 1 Levi-flat orbit is foliated by complex curves equivalent to Δ; (iii) if the codimension of $O(p)$ is 2, then $O(p)$ is either a complex curve in M or totally real. If every orbit in M has codimension 2, it can be proved by arguing as before that M is holomorphically equivalent to a product $\Delta \times S$, where S is a hyperbolic Riemann surface with $d(S) = 0$. However, in contrast with the case $d(M) = n^2 - 1$, $n \geq 3$, orbits of codimension 1, as well as totally real orbits of codimension 2, may indeed be present in a (2,3)-manifold (see Sect. 5.1 for numerous examples). Moreover, dealing with codimension 1 orbits is harder in this case than in the previously considered situations since strongly pseudoconvex orbits may be non-spherical and since the techniques that we used for working with Levi-flat orbits in the proof of Proposition 3.5 no longer apply to (2,3)-manifolds. These circumstances make the case of (2,3)-manifolds the hardest of all considered in the book.

Luckily, all 3-dimensional strongly pseudoconvex CR-homogeneous CR-manifolds were classified by E. Cartan in [CaE1] (the classification is reproduced in the proof of Theorem 5.2). Hence every codimension 1 strongly pseudoconvex orbit $O(p)$ is CR-equivalent to a manifold on E. Cartan's list, and by analyzing the list one can also determine the action of $G(M)$ on $O(p)$ (see Proposition 5.3). This allows us to classify in Theorem 5.2 all (2,3)-manifolds for which every orbit is strongly pseudoconvex.

Further, Levi-flat orbits together with all possible actions of $G(M)$ on them are classified in Proposition 5.4. Analogously to the statement of Proposition 3.5, it turns out that every Levi-flat orbit is CR-equivalent to either $\Delta \times S^1$ or $\Delta \times \mathbb{R}$. However, the action of $G(M)$ on such orbits can be very different from the actions of the "direct-product" type that occur in the statement of Proposition 3.5. Proposition 5.4 is one of the main results in this book and leads to a classification of all (2,3)-manifolds for which every orbit has codimension 1 and at least one orbit is Levi-flat (see Theorem 5.6).

Finally, we study how (at most two) codimension 2 orbits can be attached to the manifolds found in Theorems 5.2 and 5.6. This is done by considering, for every point p_0 in a codimension 2 orbit O, complex $I^0_{p_0}$-invariant curves that approach O at p_0 (cf. (1.15)). It turns out, as before, that in fact at most one codimension 2 orbit can occur. All possible manifolds with a codimension 2 orbit are listed in Theorem 5.7. The complete classification of all (2,3)-manifolds with codimension 1 orbits is summarized in Sect. 5.1.

We will now sketch the proof of Theorem 1.7.

Proof of Theorem 1.7: First of all, we remark that there is a complete C^∞-smooth K-invariant Hermitian metric h on $\mathcal{M}$. Such a metric exists not only for actions of compact groups, but also for proper actions (see [Pa], [Alek]). Let d_h the distance function on $\mathcal{M}$ induced by h. The main component of the proof is the following proposition.

Proposition 1.8. ([GK])*For every $r > 0$ the metric ball*

$$B(p,r) := \{q \in \mathcal{M} : d_h(p,q) < r\}$$

is holomorphically equivalent to $\mathbb{B}^k$ by means of a map F such that:

(i) $F(p) = 0$;

(ii) F maps $B(p,r')$ onto a ball in $\mathbb{C}^k$ centered at 0 for every $r' < r$.

We will first show how Theorem 1.7 follows from Proposition 1.8. Denote by $\mathbb{B}^k_\rho$ the ball in $\mathbb{C}^k$ of radius ρ centered at the origin. Fix an increasing sequence $\{r_m\}$ of positive numbers converging to ∞, and choose for every $m \in \mathbb{N}$ a biholomorphic map F_m from $B(p,r_m)$ onto $\mathbb{B}^k$ as in the statement of Proposition 1.8. We will now modify the sequence $\{F_m\}$ into a sequence $\{\hat{F}_m\}$ such that for every $m \in \mathbb{N}$ the following holds:

(a) $\hat{F}_m(p) = 0$;

(b) $\hat{F}_m$ biholomorphically maps $B(p,r_m)$ onto $\mathbb{B}^k_{\rho_m}$ for some ρ_m;

(c) $\hat{F}_{m+1}|_{B(p,r_m)} = \hat{F}_m$.

Set $\hat{F}_1 := F_1$, fix $m \in \mathbb{N}$ and assume that $\hat{F}_l$ for $l = 1,\dots,m$ have been constructed. We will now define $\hat{F}_{m+1}$. Clearly, there is $c > 0$ such that $\tilde{F}_{m+1} := cF_{m+1}$ maps $B(p,r_m)$ onto $\mathbb{B}^k_{\rho_m}$. Then $\mathcal{F} := \tilde{F}_{m+1} \circ \hat{F}_m^{-1}$ is an element of $\mathrm{Aut}(\mathbb{B}^k_{\rho_m})$. Since $\mathcal{F}$ preserves the origin, it is a linear transformation given by a matrix $U \in U_k$. Setting $\hat{F}_{m+1} := U^{-1} \circ \tilde{F}_{m+1}$ we obtain a map as required.

Taken together, the maps $\hat{F}_m$ define a biholomorphic map $\hat{F}$ from $\mathcal{M}$ onto $W := \cup_{m=1}^\infty \mathbb{B}^k_{\rho_m}$. Observe now that W is either $\mathbb{B}^k_\rho$ for some ρ, or all of $\mathbb{C}^k$. This completes the proof of Theorem 1.7. ∎

Remark 1.9. As was observed in [BDK], for $k \geq 2$ the assumption that $\alpha_p(K)$ acts transitively on *real* directions in $T_p(\mathcal{M})$ in Theorem 1.7 can be replaced by the seemingly weaker assumption that $\alpha_p(K)$ acts transitively on *complex* directions in $T_p(\mathcal{M})$. Indeed, let S_p be the unit sphere in $T_p(\mathcal{M})$ with respect to the metric h. Clearly, $\alpha_p(K)$ acts on S_p. Consider the *Hopf map* σ from S_p to the projectivization $\mathbb{CP}(T_p(\mathcal{M})) \simeq \mathbb{CP}^{k-1}$ of $T_p(\mathcal{M})$: $\sigma(v) = \{\lambda v : \lambda \in \mathbb{C}^*\}$ for $v \in T_p(\mathcal{M})$. The group $\alpha_p(K)$ also acts on $\mathbb{CP}(T_p(\mathcal{M}))$, and σ is $\alpha_p(K)$-equivariant.

Assume now that $\alpha_p(K)$ acts transitively on complex directions, but does not act transitively on real directions in $T_p(\mathcal{M})$. Then its action on S_p is transversal to the fibers of σ and therefore gives rise to a continuous section of σ. It is well-known, however, that σ does not have any continuous sections if $k \geq 2$ (see e.g. [NX]).

We will now give a sketch of proof of Proposition 1.8.

Proof of Proposition 1.8: First of all, it can be shown by standard methods of Riemannian geometry that, since h is complete and $\mathcal{M}$ is non-compact, the exponential map $\exp_p : T_p(\mathcal{M}) \to \mathcal{M}$ is a diffeomorphism (see Lemma 1.5 in [GK]). This fact will be essential for many arguments that appear below.

Next, let S be the set of all positive numbers r for which there exists a biholomorphic map as required in the statement of the proposition. We will show that:

(1) $S \neq \emptyset$;

(2) if for some $r > 0$ we have $(0, r) \subset S$, then $r \in S$;

(3) for every $r \in S$ there exists $\varepsilon > 0$ such that $[r, r + \varepsilon) \subset S$.

Clearly, properties (1), (2), (3) imply that $S = \mathbb{R}_+$, and the proposition follows.

To prove (1) we choose coordinates in $T_p(\mathcal{M})$ so that $\alpha_p(K) \subset U_k$ and use them to identify $T_p(\mathcal{M})$ with $\mathbb{C}^k$. Since $\alpha_p(K)$ acts transitively on real directions in $T_p(\mathcal{M})$, it acts transitively on every sphere centered at the origin. Therefore, every $\alpha_p(K)$-invariant Hermitian metric on every $\alpha_p(K)$-invariant neighborhood of the origin in $T_p(\mathcal{M})$ is proportional to the standard metric on $T_p(\mathcal{M})$. By Bochner's linearization theorem (see [Bo]) there exist a local holomorphic change of coordinates F near p on $\mathcal{M}$ that identifies a K-invariant neighborhood U of p with an $\alpha_p(K)$-invariant neighborhood $\mathcal{U}$ of the origin in $T_p(\mathcal{M})$ such that $F(p) = 0$ and $F(gs) = \alpha_p(g)F(s)$ for all $g \in K$ and $s \in U$. Since the push-forward of the Hermitian metric $h|_U$ to $\mathcal{U}$ by means of F is proportional to the standard metric on $\mathcal{U}$, we obtain that every metric ball $B(p, r) \subset U$ is mapped by F biholomorphically onto a ball in $T_p(\mathcal{M}) = \mathbb{C}^k$. Clearly, F is a map that satisfies conditions (i) and (ii), and (1) follows.

To prove (2), we argue as in the proof of Theorem 1.7: choose an increasing sequence of positive numbers $\{r_m\}$ converging to r and produce a sequence of maps $\{\hat{F}_m\}$ satisfying conditions (a)–(c) above. Then the resulting map $\hat{F}$ is a biholomorphism from $B(p, r)$ onto the corresponding set W. It then follows that $r \in S$, provided $W \neq \mathbb{C}^k$. The claim $W \neq \mathbb{C}^k$ is a consequence of certain estimates for the radial derivatives of the map $\hat{F}$. We refer the reader to Lemma 4.1 of [GK] for these technical details.

The proof of (3) is more difficult, and we will only indicate it. The main ingredient of the argument is the following lemma.

Lemma 1.10. ([GK])*If $r \in S$, then $\partial B(p,r)$ is a C^∞-smooth strongly pseudoconvex hypersurface in $\mathcal{M}$.*

Lemma 1.10 is proved by estimating higher-order derivatives of the map $F : B(p,r) \to \mathbb{B}^k$ satisfying (i) and (ii), using the method utilized in the proof of Lemma 4.1 of [GK] (see Lemma 4.2 therein). These estimates give that F extends to a C^∞-smooth map from $\overline{B(p,r)}$ to $\overline{\mathbb{B}^k}$ and F^{-1} extends to a C^∞-smooth map from $\overline{\mathbb{B}^k}$ to $\overline{B(p,r)}$.

It follows from Lemma 1.10 that for some $\varepsilon > 0$ the set $\partial B(p,R)$ is a C^∞-smooth strongly pseudoconvex hypersurface for every $R \in [r, r+\varepsilon)$. This immediately yields that for all such R the metric ball $B(p,R)$ is a strongly pseudoconvex Stein manifold. Let $F : B(p,r) \to \mathbb{B}^k$ be a biholomorphism satisfying (i) and (ii), and f a diffeomorphism from $\overline{B(p,R)}$ onto $\overline{B(p,r)}$ that commutes with the action of K and that is close to the identity in the C^∞-topology on $\overline{B(p,R)}$. Denote by F_j the components of the map F and consider the functions $F_j \circ f$, $j = 1, \dots, k$. Since $\overline{\partial}(F_j \circ f)$ is C^∞-small on $\overline{B(p,R)}$, one can find a solution g_j to the equation

$$\overline{\partial} g_j = \overline{\partial}(F_j \circ f),$$

such that g_j is orthogonal to holomorphic functions on $B(p,R)$ with respect to the Hermitian metric h, and such that g_j is C^1-close to 0 on $\overline{B(p,R)}$, $j = 1, \dots, k$. For every j set $\hat{F}_j := F_j \circ f - g_j$ and define $\hat{F} := (\hat{F}_1, \dots, \hat{F}_k)$. It can be shown that if ε is sufficiently small, then, for a linear transformation A of $\mathbb{C}^k$, the composition $A \circ \hat{F}$ is a biholomorphism from $B(p,R)$ onto $\mathbb{B}^k$ satisfying (i) and (ii). This finishes the proof of Proposition 1.8. ■

We close this section with an analogue of Theorem 1.7 for compact manifolds. It is not relevant to our study of hyperbolic manifolds and is only referred to in Chap. 6, where proper group actions on not necessarily hyperbolic manifolds are briefly discussed.

Theorem 1.11. ([BDK])*Let $\mathcal{M}$ be a connected compact complex manifold of dimension $k \geq 2$ and p a point in $\mathcal{M}$. Let $K \subset Aut(\mathcal{M})$ be a compact subgroup that fixes p and assume that the group $\alpha_p(K)$ acts transitively on complex directions in $T_p(\mathcal{M})$. Then $\mathcal{M}$ is holomorphically equivalent to $\mathbb{CP}^k$.*

Observe that Remark 1.9 applies to compact manifolds as well, and therefore the condition that $\alpha_p(K)$ acts transitively on complex directions in $T_p(\mathcal{M})$ in Theorem 1.11 is equivalent to the condition that it acts transitively on real directions in $T_p(\mathcal{M})$.

The idea of the proof of Theorem 1.11 is to show that $\mathcal{M}$ is obtained from $\mathbb{C}^k$ by attaching a copy of $\mathbb{CP}^{k-1}$. If a complex manifold has this structure, it is holomorphically equivalent to $\mathbb{CP}^k$ (see [BreM]). As in the proof of Theorem 1.7, the manifold $\mathcal{M}$ can be equipped with a C^∞-smooth K-invariant Hermitian metric. Let C_p be the cut locus at p. The aim of the arguments in [BDK] is to show that $\mathcal{M} \setminus C_p$ is holomorphically equivalent to $\mathbb{C}^k$ and that C_p is a complex hypersurface holomorphically equivalent to $\mathbb{CP}^{k-1}$. We refer the interested reader to [BDK] for details.

2

The Homogeneous Case

In this chapter we describe all connected homogeneous hyperbolic manifolds with $n^2 - 1 \leq d(M) \leq n^2 + 2$.

2.1 Homogeneity for $d(M) > n^2$

We begin by showing that for $d(M) = n^2 + 1$ and $d(M) = n^2 + 2$ only homogeneous manifolds can occur.

Proposition 2.1. ([Ka]) *Let M be a connected hyperbolic manifold of dimension n for which $d(M) > n^2$. Then M is homogeneous.*

Proof: Fix $p \in M$ and let $V \subset T_p(M)$ be the tangent space to $O(p)$ at p. Clearly, V is L_p-invariant. We assume now that $V \neq T_p(M)$ and consider the following three cases.

Case 1. $d := \dim_{\mathbb{C}}(V + iV) < n$.

The natural representation of L_p on $T_p(M)$ is completely reducible and the subspace $V + iV$ is L_p-invariant. Hence in some coordinates in $T_p(M)$ the group L_p becomes a subgroup of $U_{n-d} \times U_d$. Since $\dim O(p) \leq 2d$, it follows that

$$d(M) \leq (n-d)^2 + d^2 + 2d.$$

Observe, however, that this is impossible since the right-hand side in the above inequality does not exceed n^2.

Case 2. $T_p(M) = V + iV$ and $r := \dim_{\mathbb{C}}(V \cap iV) > 0$.

As above, L_p can be viewed as a subgroup of $U_{n-r} \times U_r$ (observe that $r < n$). Moreover, $V \cap iV \neq V$ and, since L_p preserves V, it follows that $\dim L_p < r^2 + (n-r)^2$. We have $\dim O(p) \leq 2n - 1$, and therefore

$$d(M) < (n-r)^2 + r^2 + 2n - 1.$$

Observe now that the right-hand side does not exceed $n^2 + 1$ which gives a contradiction.

Case 3. $T_p(M) = V \oplus iV$.

In this case $\dim V = n$, and L_p can be viewed as a subgroup of $O_n(\mathbb{R})$, which yields

$$d(M) \leq \frac{n(n-1)}{2} + n.$$

Again, this is impossible since the right-hand side does not exceed $n^2 - 1$.

Thus, we have shown that $V = T_p(M)$ for every $p \in M$. Therefore, M is homogeneous, and the proof of the proposition is complete. ■

Observe that if $d(M) \leq n^2$, the manifold M need not be homogeneous. Indeed, for any spherical shell

$$S_r := \{z \in \mathbb{C}^n : r < |z| < 1\}, \tag{2.1}$$

with $0 \leq r < 1$, the group $\mathrm{Aut}(S_r)$ coincides with U_n and thus has dimension n^2, but the action of U_n on S_r is not transitive.

2.2 Classification of Homogeneous Manifolds

In this section we give a complete classification of all connected homogeneous n-dimensional hyperbolic manifolds with $n^2 - 1 \leq d(M) \leq n^2 + 2$. Due to Proposition 2.1 this classification implies a full description of manifolds with $d(M) = n^2 + 1$ and $d(M) = n^2 + 2$.

Theorem 2.2. ([IKra2], [I2], [I3]) *Let M be a connected homogeneous hyperbolic manifold of dimension $n \geq 2$, with $n^2 - 1 \leq d(M) \leq n^2 + 2$. Then the following holds:*

(i) if $d(M) = n^2 + 2$, then M is holomorphically equivalent to $\mathbb{B}^{n-1} \times \Delta$;

(ii) if $d(M) = n^2 + 1$, then $n = 3$ and M is holomorphically equivalent to the Siegel space

$$\mathscr{S} := \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 : Z\overline{Z} \ll \mathrm{id}\right\},$$

where

$$Z := \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix};$$

(iii) if $d(M) = n^2$, then either $n = 3$ and M is holomorphically equivalent to Δ^3, or $n = 4$ and M is holomorphically equivalent to $\mathbb{B}^2 \times \mathbb{B}^2$;

(iv) if $d(M) = n^2 - 1$, then $n = 4$ and M is holomorphically equivalent to the tube domain

$$\mathscr{T} := \Big\{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : (Im\, z_1)^2 + (Im\, z_2)^2 + (Im\, z_3)^2 - (Im\, z_4)^2 < 0,\ Im\, z_4 > 0\Big\}.$$

Before proving Theorem 2.2 we remark that $\mathscr{S}$ is the symmetric classical domain of type (III_2) with $\mathrm{Aut}(\mathscr{S}) \simeq Sp_4(\mathbb{R})/\mathbb{Z}_2$, and $\mathscr{T}$ is the symmetric classical domain of type ($\mathrm{I}_{2,2}$) with $G(\mathscr{T}) \simeq SU_{2,2}/\mathbb{Z}_4$ (see [Sa]).[1]

Proof of Theorem 2.2: Since M is homogeneous, by the results of [N], [P-S], it is holomorphically equivalent to a Siegel domain $D \subset \mathbb{C}^n$ (see (1.6)). Further, since M cannot be equivalent to $\mathbb{B}^n$, we have $k \geq 2$. For $n = 2$ we then immediately see that M is holomorphically equivalent to Δ^2 (cf. [CaE2]), which is a special case of (i) of the theorem. For $n = 3$ it follows that M is equivalent to one of the following domains: $\mathbb{B}^2 \times \Delta$, Δ^3, $\mathscr{S}$ (observe that $d(\mathbb{B}^2 \times \Delta) = 11 = 3^2 + 2$, $d(\mathscr{S}) = 10 = 3^2 + 1$, $d(\Delta^3) = 9 = 3^2$). These domains are listed in (i), (ii) and (iii) of the theorem, respectively.

Suppose now that $n \geq 4$. It follows from (1.7)–(1.9) that

$$d(D) \leq 4n - 2k + \dim \mathfrak{g}_0(D). \tag{2.2}$$

Clearly, in (1.11) the matrix A is determined by the matrix B up to a matrix $L \in \mathfrak{gl}_{n-k}$ satisfying

$$F(Lz, z) + F(z, Lz) = 0,$$

for all $z \in \mathbb{C}^{n-k}$. Such matrices L form a real linear subspace $\mathcal{L}$ of $\mathfrak{gl}_{n-k}$, and we denote by s its dimension. Then (1.10) implies

$$\dim \mathfrak{g}_0(D) \leq s + \dim \mathfrak{g}_{\mathrm{lin}}(C). \tag{2.3}$$

By the definition of Siegel domain, there exists a positive-definite linear combination H of the components of the Hermitian form F. Hence $\mathcal{L}$ is a subspace of the Lie algebra of matrices skew-Hermitian with respect to H and thus $s \leq (n-k)^2$. Therefore, inequalities (2.2) and (2.3) imply

$$d(D) \leq 4n - 2k + (n-k)^2 + \dim \mathfrak{g}_{\mathrm{lin}}(C). \tag{2.4}$$

We now need the following lemma.

Lemma 2.3. *We have*

$$\dim \mathfrak{g}_{\mathit{lin}}(C) \leq \frac{k^2}{2} - \frac{k}{2} + 1. \tag{2.5}$$

[1] The group $Sp_4(\mathbb{R})$ is a non-compact real form of $Sp_4(\mathbb{C})$ and should not be confused with the compact real form Sp_2 introduced earlier.

Proof of Lemma 2.3: Fix a point $x_0 \in C$ and consider its isotropy subgroup $G_{x_0} \subset G_{\rm lin}(C)$. This subgroup is compact since the bounded open set $C \cap (x_0 - C)$ is G_{x_0}-invariant. Then, changing coordinates in $\mathbb{R}^k$ if necessary, we can assume that G_{x_0} is contained in $O_k(\mathbb{R})$. The group $O_k(\mathbb{R})$ acts transitively on the sphere of radius $|x_0|$ in $\mathbb{R}^k$, and the isotropy subgroup H_{x_0} of x_0 under the $O_k(\mathbb{R})$-action is isomorphic to $O_{k-1}(\mathbb{R})$. Since $G_{x_0} \subset H_{x_0}$, we have

$$\dim G_{x_0} \le \dim H_{x_0} = \frac{k^2}{2} - \frac{3k}{2} + 1,$$

which implies inequality (2.5), as required. ■

Inequalities (2.4) and (2.5) yield

$$d(D) \le \frac{3k^2}{2} - k\left(2n + \frac{5}{2}\right) + n^2 + 4n + 1. \tag{2.6}$$

It is straightforward to verify that the right-hand side of (2.6) is strictly less than n^2 for $n \ge 4$ and $k \ge 3$.

Assume first that $n^2 \le d(M) \le n^2 + 2$. It then follows that $k = 2$, and therefore $\dim \mathfrak{g}_{\rm lin}(C) = 2$. Without loss of generality we can assume that the first component F_1 of the $\mathbb{C}^2$-valued Hermitian form F is positive-definite. Furthermore, applying an appropriate linear transformation of the z-variables, we can assume that F_1 is given by the identity matrix and F_2 by a diagonal matrix.

Suppose now that $d(M)$ is either n^2+1 or n^2+2. We will then show that the matrix of F_2 is scalar. Indeed, assuming that this matrix has a pair of distinct eigenvalues, we see that $s \le (n-2)^2 - 2$, and therefore (2.2), (2.3) yield $d(D) \le n^2$, which contradicts our assumption. Thus, F_2 is scalar, and therefore D is holomorphically equivalent to one of the following domains:

$$\begin{aligned} D_1 &:= \left\{(z,w) \in \mathbb{C}^{n-2} \times \mathbb{C}^2 : \operatorname{Im} w_1 - |z|^2 > 0,\ \operatorname{Im} w_2 > 0\right\}, \\ D_2 &:= \left\{(z,w) \in \mathbb{C}^{n-2} \times \mathbb{C}^2 : \operatorname{Im} w_1 - |z|^2 > 0,\ \operatorname{Im} w_2 - |z|^2 > 0\right\}. \end{aligned}$$

The domain D_1 is equivalent to $\mathbb{B}^{n-1} \times \Delta$. We will now show that $d(D_2) < n^2 - 1$. We have $\dim \mathfrak{g}_{-1}(D_2) = 2$, $\dim \mathfrak{g}_{-1/2}(D_2) = 2(n-2)$ (see (1.8), (1.9)), and it is clear from (2.3) that $\dim \mathfrak{g}_0(D_2) \le n^2 - 4n + 6$ (note that in this case $C = \{(x,y) \in \mathbb{R}^2 : x > 0,\ y > 0\}$). Further, the explicit descriptions of $\mathfrak{g}_{1/2}(D_2)$ and $\mathfrak{g}_1(D_2)$ (see [Sa]) imply that both these components are trivial. Therefore

$$d(D_2) \le n^2 - 2n + 4 < n^2 - 1,$$

hence D cannot be equivalent to D_2 (alternatively, to exclude D_2 we could observe that it is not homogeneous).

Thus, we have shown that if $d(M)$ is either n^2+1 or n^2+2, then M is holomorphically equivalent to $\mathbb{B}^{n-1} \times \Delta$, which is listed in (i) of the theorem.

Suppose now that $d(M) = n^2$. It then follows from (2.2), (2.3) that

$$s \geq n^2 - 4n + 2. \tag{2.7}$$

If the matrix of F_2 is scalar, then by our argument above, D is equivalent to either $\mathbb{B}^{n-1} \times \Delta$ or D_2, which is impossible. Thus, the matrix of F_2 is not scalar, and inequality (2.7) yields that it must have exactly one pair of distinct eigenvalues. Therefore $n = 4$, $s = 2$, and we have

$$\dim \mathfrak{g}_{1/2}(D) = 4, \quad \dim \mathfrak{g}_1(D) = 2.$$

It now follows from the explicit descriptions of $\mathfrak{g}_{1/2}(D)$ and $\mathfrak{g}_1(D)$ (see [Sa]) that D is holomorphically equivalent to $\mathbb{B}^2 \times \mathbb{B}^2$, which is listed in (iii) of the theorem.

Suppose finally that $d(M) = n^2 - 1$. Observe that the right-hand side of (2.6) is strictly less than $n^2 - 1$ for $n \geq 5$ and $k \geq 3$, and is greater than or equal to 15 for $n = 4$. Furthermore, for $n = 4$ the right-hand side of (2.6) is equal to 15 only if $k = 3$ or $k = 4$, and it follows from (2.4), (2.5) that in these cases $\dim \mathfrak{g}_{\text{lin}}(C) = k^2/2 - k/2 + 1$.

Assume first that $n = 4$ and the right-hand side of (2.6) is equal to 15. Then C is $G_{\text{lin}}(C)$-homogeneous and for every point $x_0 \in C$ there exist coordinates in $\mathbb{R}^k$ such that the isotropy subgroup G_{x_0} contains $SO_{k-1}(\mathbb{R})$ (see the proof of Lemma 2.3). Then after a linear change of coordinates the cone C takes the form

$$\left\{(x_1, \ldots, x_k) \in \mathbb{R}^k : [x, x] < 0,\, x_k > 0\right\},$$

where $x := (x_1, \ldots, x_k)$ and $[x, x] := x_1^2 + \ldots + x_{k-1}^2 - x_k^2$. In these coordinates the algebra $\mathfrak{g}_{\text{lin}}(C)$ is generated by the center of $\mathfrak{gl}_k(\mathbb{R})$ and the algebra of pseudo-orthogonal matrices $\mathfrak{o}_{k-1,1}(\mathbb{R})$. Suppose that $k = 3$. Then we have $F = (v_1|z|^2, v_2|z|^2, v_3|z|^2)$ for some vector $v := (v_1, v_2, v_3) \in \overline{C} \setminus \{0\}$. It follows from (1.11) that v is an eigenvector of the matrix B for every element of $\mathfrak{g}_0(D)$, which implies that $\dim \mathfrak{g}_0(D) \leq 4$. Hence by (2.2) we have $d(D) \leq 14$, which is impossible. Next, if $k = 4$, then D is holomorphically equivalent to the tube domain $\mathscr{T}$ listed in (iv) of the theorem.

Assume now that $n \geq 4$ and $k = 2$. Then (2.2), (2.3) imply

$$s \geq n^2 - 4n + 1. \tag{2.8}$$

Furthermore, applying an appropriate linear transformation, we can assume that F_1 is given by the identity matrix and F_2 by a diagonal matrix. If the matrix of F_2 is scalar, then we obtain, as before, that D is equivalent to either $\mathbb{B}^{n-1} \times \Delta$ or D_2, which is impossible. Thus, the matrix of F_2 is not scalar, and inequality (2.8) yields that it must have exactly one pair of distinct eigenvalues. Therefore $n = 4$, $s = 2$, and we have

$$\dim \mathfrak{g}_{1/2}(D) + \dim \mathfrak{g}_1(D) \geq 5.$$

It now follows from the explicit descriptions of $\mathfrak{g}_{1/2}(D)$ and $\mathfrak{g}_1(D)$ (see [Sa]) that D is holomorphically equivalent to $\mathbb{B}^2 \times \mathbb{B}^2$. This is clearly impossible, and the proof of the theorem is complete. ∎

Remark 2.4. Using Proposition 2.1 and arguing as above, one can also obtain an alternative proof of Theorem 1.3.

The following corollary is an immediate consequence of Proposition 2.1 and Theorem 2.2.

Corollary 2.5. ([IKra2]) *Let M be a connected hyperbolic manifold of dimension $n \geq 2$, with either $d(M) = n^2 + 2$ or $d(M) = n^2 + 1$. Then M is holomorphically equivalent to either $\mathbb{B}^{n-1} \times \Delta$ or the Siegel space $\mathscr{S}$, respectively.*

In the following three chapters we treat the non-homogeneous case which gives the vast majority of manifolds in our classification.

3

The Case $d(M) = n^2$

In this chapter we give an explicit classification of connected non-homogeneous hyperbolic manifolds of dimension $n \geq 2$ with $d(M) = n^2$. The classification is summarized in Theorem 3.1. We start by making introductory remarks and formulating our result.

3.1 Main Result

In [GIK] we classified all hyperbolic Reinhardt domains in $\mathbb{C}^n$ with $d(M) = n^2$. That classification was based on the description of the automorphism group of a hyperbolic Reinhardt domain obtained in [Kru]. Further, in [KV] simply connected *complete* hyperbolic manifolds with $d(M) = n^2$ were studied (a hyperbolic manifold is called complete if the Kobayashi distance on the manifold is complete). The main result of [KV] states that every such manifold is holomorphically equivalent to a Reinhardt domain and hence the classification in this case is a subset of the classification in [GIK]. There are, however, examples of hyperbolic manifolds with $d(M) = n^2$ outside the class of Reinhardt domains. For instance, the quotient of a spherical shell $S_r/\mathbb{Z}_m$ (see (2.1)), where $\mathbb{Z}_m$ is realized as a subgroup of scalar matrices in U_n, is not equivalent to any Reinhardt domain if $m > 1$, but has the n^2-dimensional automorphism group $U_n/\mathbb{Z}_m$. Further, not all hyperbolic manifolds with $d(M) = n^2$ are complete and simply connected. Consider, for example, the Reinhardt domains $\mathcal{E}_{r,\theta} := \Big\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'| < 1,\, r(1 - |z'|^2)^\theta < |z_n| < (1 - |z'|^2)^\theta\Big\}$, with either $\theta \geq 0$, $0 \leq r < 1$, or $\theta < 0$, $r = 0$. None of them is simply connected, and complete domains only arise for either $\theta = 0$, or $r = 0$, $\theta > 0$. At the same time, each of these domains has an n^2-dimensional automorphism group. This group consists of the maps

$$\begin{aligned} z' &\mapsto \frac{Az' + b}{cz' + d}, \\ z_n &\mapsto \frac{e^{i\beta} z_n}{(cz' + d)^{2\theta}}, \end{aligned} \tag{3.1}$$

where

$$\begin{pmatrix} A & b \\ c & d \end{pmatrix} \in SU_{n-1,1}, \quad \beta \in \mathbb{R}.$$

In this chapter we obtain a complete classification of hyperbolic manifolds with $d(M) = n^2$ without any additional assumptions. We will now state our main result.

Theorem 3.1. ([I2]) *Let M be a connected non-homogeneous hyperbolic manifold of dimension $n \geq 2$ with $d(M) = n^2$. Then M is holomorphically equivalent to one of the following manifolds:*

$$\begin{aligned}
&\textit{(i)} \quad S_r/\mathbb{Z}_m,\ 0 \leq r < 1,\ m \in \mathbb{N}; \\
&\textit{(ii)} \quad \mathfrak{E}_\theta := \Big\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'|^2 + |z_n|^\theta < 1\Big\}, \\
&\qquad \theta > 0,\ \theta \neq 2; \\
&\textit{(iii)} \quad \mathcal{E}_\theta := \Big\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'| < 1,\ |z_n| < \\
&\qquad (1 - |z'|^2)^\theta\Big\},\ \theta < 0; \\
&\textit{(iv)} \quad \mathcal{E}_{r,\theta} = \Big\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'| < 1,\ r(1 - |z'|^2)^\theta < \\
&\qquad |z_n| < (1 - |z'|^2)^\theta\Big\},\ \textit{with either } \theta \geq 0,\ 0 \leq r < 1, \\
&\qquad \textit{or } \theta < 0,\ r = 0; \\
&\textit{(v)} \quad \mathfrak{A}_{r,\theta} := \Big\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : r \exp\big(\theta|z'|^2\big) < |z_n| < \\
&\qquad \exp\big(\theta|z'|^2\big)\Big\},\ \textit{with either } \theta = 1,\ 0 < r < 1, \\
&\qquad \textit{or } \theta = -1,\ r = 0; \\
&\textit{(vi)} \quad \mathfrak{B}_{r,\theta} := \Big\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'| < 1,\ r(1 - |z'|^2)^\theta < \\
&\qquad \exp(\mathrm{Re}\, z_n) < (1 - |z'|^2)^\theta\Big\},\ \textit{with either} \\
&\qquad \theta = 1,\ 0 \leq r < 1 \textit{ or } \theta = -1,\ r = 0; \\
&\textit{(vii)} \quad \mathfrak{C} := \Big\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : -1 + |z'|^2 < \mathrm{Re}\, z_n < |z'|^2\Big\}.
\end{aligned} \tag{3.2}$$

The manifolds on list (3.2) are pairwise holomorphically non-equivalent.

The n^2-dimensional automorphism groups of manifolds (3.2) are not hard to find and will explicitly appear during the course of proof of Theorem 3.1. Except those mentioned earlier, they are as follows: $\mathrm{Aut}(\mathfrak{E}_\theta)$ is obtained from formula (3.1) by replacing θ with $1/\theta$, $\mathrm{Aut}(\mathfrak{A}_{0,-1})$ consists of all maps of the form

$$\begin{aligned} z' &\mapsto Uz' + a, \\ z_n &\mapsto e^{i\beta} \exp\Big(-2\langle Uz', a\rangle - |a|^2\Big) z_n, \end{aligned}$$

where $U \in U_{n-1}$, $a \in \mathbb{C}^{n-1}$, $\beta \in \mathbb{R}$, and $\langle \cdot\,, \cdot\rangle$ denotes the inner product in $\mathbb{C}^{n-1}$; $\mathrm{Aut}(\mathfrak{A}_{r,1})$ is given by (3.13), $\mathrm{Aut}(\mathfrak{B}_{r,\theta})$ is given by (3.11), $\mathrm{Aut}(\mathfrak{C})$ consists of all maps of the form (3.7) with $\lambda = 1$.

The domains S_r, $\mathfrak{E}_\theta$, $\mathcal{E}_\theta$, $\mathcal{E}_{r,\theta}$, $\mathfrak{A}_{r,\theta}$, (as well as the homogeneous domains Δ^3 and $\mathbb{B}^2 \times \mathbb{B}^2$) are the Reinhardt domains from the classification in [GIK]. Each of the domains $\mathfrak{B}_{r,\theta}$ is the universal cover of some $\mathcal{E}_{r',\theta'}$, namely, $\mathfrak{B}_{r^{1/\theta},1}$ covers $\mathcal{E}_{r,\theta}$ for $0 \le r < 1$, $\theta > 0$, and $\mathfrak{B}_{0,-1}$ covers $\mathcal{E}_{0,\theta}$ for $\theta < 0$. Note that the universal cover of $\mathcal{E}_{r,0}$ for every $0 \le r < 1$ is $\mathbb{B}^{n-1} \times \Delta$ and hence has an automorphism group of dimension n^2+2. Observe also that $\mathfrak{C}$ is the universal cover of the domain $\mathfrak{A}_{r,1}$ for every $0 < r < 1$, and that the universal cover of $\mathfrak{A}_{0,-1}$ is $\mathbb{B}^n$ and thus has an automorphism group of dimension $n^2 + 2n$.

The only manifolds in (3.2) that are both simply connected and complete hyperbolic are $\mathfrak{E}_\theta$ listed under (ii). Together with the homogeneous domains Δ^3 and $\mathbb{B}^2 \times \mathbb{B}^2$, they form the partial classification obtained in [KV].

The chapter is organized as follows. In Sect. 3.2 we give an initial classification of the orbits of the $G(M)$-action on M. It turns out that every $G(M)$-orbit is either a real or complex hypersurface in M. Furthermore, real hypersurface orbits are either spherical or Levi-flat, and in the latter case they are foliated by complex submanifolds holomorphically equivalent to the ball $\mathbb{B}^{n-1}$; there are at most two complex hypersurface orbits and each of them is holomorphically equivalent to $\mathbb{B}^{n-1}$ (see Proposition 3.2).

Next, in Sect. 3.3 we determine real hypersurface orbits up to CR-equivalence. In the spherical case there are five possible kinds of orbits (see Proposition 3.3). In the Levi-flat case every orbit is shown to be equivalent to either $\mathbb{B}^{n-1} \times \mathbb{R}$ or $\mathbb{B}^{n-1} \times S^1$ (see Proposition 3.5). The presence of an orbit of a particular kind determines $G(M)$ as a Lie group (see Propositions 3.4 and 3.5).

In Sect. 3.4 we prove Theorem 3.1 by studying how orbits of the kinds found in Sect. 3.3 can be glued together and obtain the manifolds on list (3.2). In particular, it turns out that Levi-flat orbits equivalent to $\mathbb{B}^{n-1} \times \mathbb{R}$ cannot occur, since they lead to $\mathbb{B}^{n-1} \times \Delta$, and that there can be in fact at most one complex hypersurface orbit.

3.2 Initial Classification of Orbits

In this section we prove the following proposition.

Proposition 3.2. ([I2])*Let M be a connected non-homogeneous hyperbolic manifold of dimension $n \geq 2$ with $d(M) = n^2$. Fix $p \in M$ and let $V := T_p(O(p))$. Then*

(i) the orbit $O(p)$ is a real or complex closed hypersurface in M;

(ii) if $O(p)$ is a real hypersurface, it is either spherical, or Levi-flat and foliated by submanifolds holomorphically equivalent to $\mathbb{B}^{n-1}$; there exist coordinates in $T_p(M)$ such that – with respect to the orthogonal decomposition $T_p(M) = (V \cap iV)^{\perp} \oplus (V \cap iV)$ – the group L_p is either $\{id\} \times U_{n-1}$ or $\mathbb{Z}_2 \times U_{n-1}$, and the latter can only occur if $O(p)$ is Levi-flat;

(iii) if $O(p)$ is a complex hypersurface, it is holomorphically equivalent to $\mathbb{B}^{n-1}$; there exist coordinates in $T_p(M)$ such that – with respect to the orthogonal decomposition $T_p(M) = V^{\perp} \oplus V$ – we have $L_p = U_1 \times U_{n-1}$, the subgroup $I'_p := \alpha_p^{-1}(U_1)$ (where α_p is the isotropy representation at p – see (1.2)) is normal in $G(M)$, and the quotient $G(M)/I'_p$ is isomorphic to $Aut(\mathbb{B}^{n-1})$; there are at most two complex hypersurface orbits in M.

Proof: We consider the three cases as in the proof of Proposition 2.1. Below we use the notation introduced therein.

Case 1. $d := \dim_{\mathbb{C}}(V + iV) < n$.

Arguing as in the proof of Proposition 2.1 we obtain

$$n^2 \leq (n-d)^2 + d^2 + 2d,$$

and therefore we have either $d = 0$ or $d = n - 1$. If $d = 0$, then $O(p) = \{p\}$. Now Folgerung 1.10 of [Ka] implies that M is holomorphically equivalent to $\mathbb{B}^n$, which is impossible, and thus $d = n - 1$ (for an alternative proof of this fact see Lemma 3.1 of [KV]). Furthermore, we have

$$n^2 = \dim L_p + \dim O(p) \leq n^2 - 2n + 2 + \dim O(p).$$

Hence $\dim O(p) \geq 2n - 2$ which implies that $\dim O(p) = 2d = 2n - 2$, and therefore $iV = V$, which means that $O(p)$ is a complex closed hypersurface in M.

We have $L_p = U_1 \times U_{n-1}$. The U_{n-1}-component of L_p acts transitively on real directions in V and therefore by Theorem 1.7 the orbit $O(p)$ is holomorphically equivalent to $\mathbb{B}^{n-1}$. On the other hand, the U_1-component of L_p acts trivially on V and therefore the subgroup $I'_p = \alpha_p^{-1}(U_1)$ of I_p corresponding to this component is the kernel of the action of $G(M)$ on $O(p)$ (this follows from [Bo], see also [Ka]). Thus, I'_p is normal in $G(M)$ and the quotient $G(M)/I'_p$ acts effectively on $O(p)$. Since $G(M)/I'_p$ has dimension $n^2 - 1 = d(\mathbb{B}^{n-1})$, it is isomorphic to $\mathrm{Aut}(\mathbb{B}^{n-1})$.

Case 2. $T_p(M) = V + iV$ and $r := \dim_{\mathbb{C}}(V \cap iV) > 0$.

In this case we obtain

$$n^2 < (n-r)^2 + r^2 + 2n - 1,$$

which shows that either $r = 1$ or $r = n - 1$. It then follows that $\dim L_p < n^2 - 2n + 2$. Therefore, we have

$$n^2 = \dim L_p + \dim O(p) < n^2 - 2n + 2 + \dim O(p).$$

Hence $\dim O(p) > 2n - 2$ and thus $\dim O(p) = 2n - 1$. This yields that $O(p)$ is a real closed hypersurface in M, and therefore $r = n - 1$.

Let W be the orthogonal complement to $T_p^c(O(p)) = V \cap iV$ in $T_p(M)$. Since $r = n-1$, we have $\dim_{\mathbb{C}} W = 1$. The group L_p is a subgroup of U_n and preserves V, $T_p^c(O(p))$, and W; hence it preserves the line $W \cap V$. Therefore, it can act only as $\pm\mathrm{id}$ on W, that is, $L_p \subset \mathbb{Z}_2 \times U_{n-1}$. Since $\dim L_p = (n-1)^2$, we have either $L_p = \{\mathrm{id}\} \times U_{n-1}$, or $L_p = \mathbb{Z}_2 \times U_{n-1}$. In particular, L_p acts transitively on real directions in $T_p^c(O(p))$. Hence $O(p)$ is either Levi-flat or strongly pseudoconvex. If $O(p)$ is strongly pseudoconvex, we have $L_p = \{\mathrm{id}\} \times U_{n-1}$.

Suppose that $O(p)$ is Levi-flat. Then $O(p)$ is foliated by connected complex submanifolds. Consider the leaf $O(p)_p$ passing through p. Since I_p preserves $O(p)_p$ and L_p acts transitively on real directions in the tangent space $T_p^c(O(p))$ to $O(p)_p$, it follows from Theorem 1.7 that $O(p)_p$ is holomorphically equivalent to $\mathbb{B}^{n-1}$. The same argument applies to any other leaf in $O(p)$.

Assume now that $O(p)$ is strongly pseudoconvex. Since $L_p = \{\mathrm{id}\} \times U_{n-1}$, we have $d_p(O(p)) \geq (n-1)^2$, and Proposition 1.5 yields that $O(p)$ is spherical.

Case 3. $T_p(M) = V \oplus iV$.

In this case we obtain a contradiction as in the proof of Proposition 2.1.

We will now show that there can be at most two complex hypersurface orbits in M. Let $O(p)$ be a complex hypersurface for some $p \in M$. Since L_p acts as U_1 on a complement to V in $T_p(M)$ (see Case 1), there exists a neighborhood U of p such that for every $q \in U \setminus O(p)$ the values at q of $G(M)$-vector fields span a codimension 1 subspace of $T_q(M)$. Hence there is always a real hypersurface orbit in M. Since the action of $G(M)$ on M is proper, it follows that the orbit space $M/G(M)$ is homeomorphic to one of the following: $\mathbb{R}$, S^1, $[0,1]$, $[0,1)$ (see [AA]), and thus there can be at most two complex hypersurface orbits in M.

The proof of the proposition is complete. ■

3.3 Real Hypersurface Orbits

In this section we classify real hypersurface orbits up to CR-equivalence. We deal with spherical orbits first.

Proposition 3.3. ([I2])*Let M be a connected non-homogeneous hyperbolic manifold of dimension $n \geq 2$ with $d(M) = n^2$. Assume that for a point $p \in M$ its orbit $O(p)$ is spherical. Then $O(p)$ is CR-equivalent to one of the following hypersurfaces:*

$$\begin{array}{ll} (i) & \text{a lens space } \mathcal{L}_m := S^{2n-1}/\mathbb{Z}_m \text{ for some } m \in \mathbb{N};\\ (ii) & \sigma := \left\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : Re\, z_n = |z'|^2\right\};\\ (iii) & \delta := \left\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z_n| = \exp\left(|z'|^2\right)\right\};\\ (iv) & \omega := \left\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'|^2 + \exp\left(Re\, z_n\right) = 1\right\};\\ (v) & \varepsilon_\alpha := \left\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'|^2 + |z_n|^\alpha = 1,\ z_n \neq 0\right\},\\ & \qquad \text{for some } \alpha > 0. \end{array} \tag{3.3}$$

Proof: Recall that for a connected smooth strongly pseudoconvex CR-manifold N we denoted by $\mathrm{Aut}_{CR}(N)$ the group of CR-automorphisms of N equipped with the compact-open topology. It is known (see [Ta], [CM], [BS2], [Sch]) that $\mathrm{Aut}_{CR}(N)$ is a Lie group (general results regarding Lie group structures on groups of CR-automorphisms can be found in [BRWZ], for related results see also [F]). Let $\tilde{O}(p)$ be the universal cover of $O(p)$. The group $\mathrm{Aut}_{CR}(O(p))^0$ acts transitively on $O(p)$ and therefore its universal cover $\widetilde{\mathrm{Aut}}_{CR}(O(p))^0$ acts transitively on $\tilde{O}(p)$. Let G be the (possibly non-closed) subgroup of $\mathrm{Aut}_{CR}(\tilde{O}(p))$ that consists of all CR-automorphisms of $\tilde{O}(p)$ generated by this action. Observe that G is a Lie group isomorphic to the quotient of $\widetilde{\mathrm{Aut}}_{CR}(O(p))^0$ by a discrete central subgroup. Let $\varGamma \subset \mathrm{Aut}_{CR}(\tilde{O}(p))$ be the group of deck transformations associated with the covering map $\tilde{O}(p) \to O(p)$. The group $\varGamma$ acts freely properly discontinuously on $\tilde{O}(p)$, lies in the centralizer of G in $\mathrm{Aut}_{CR}(\tilde{O}(p))$ and is isomorphic to H/H^0, with $H := \pi^{-1}(J_p)$, where $\pi : \widetilde{\mathrm{Aut}}_{CR}(O(p))^0 \to \mathrm{Aut}_{CR}(O(p))^0$ is the covering map and J_p is the isotropy subgroup of p under the action of $\mathrm{Aut}_{CR}(O(p))^0$.

The manifold $\tilde{O}(p)$ is spherical, and there is a local CR-isomorphism $\varPi$ from $\tilde{O}(p)$ onto a domain $D \subset S^{2n-1}$. By Proposition 1.4 of [BS1], $\varPi$ is a covering map. Further, for every $f \in \mathrm{Aut}_{CR}(\tilde{O}(p))$ there is $g \in \mathrm{Aut}(D)$ such that

$$g \circ \varPi = \varPi \circ f, \tag{3.4}$$

that is, f is a *lift* of g under Π (observe also that every element of the group G introduced above is a lift of an element of $\mathrm{Aut}_{CR}(O(p))^0$ under the covering map $\tilde{O}(p) \to O(p)$). Since $\tilde{O}(p)$ is homogeneous, (3.4) implies that D is homogeneous as well, and $\dim \mathrm{Aut}_{CR}(\tilde{O}(p)) = \dim \mathrm{Aut}_{CR}(D)$.

Clearly, $\dim \mathrm{Aut}_{CR}(O(p)) \geq n^2$ and therefore we have $\dim \mathrm{Aut}_{CR}(D) \geq n^2$. All homogeneous domains in S^{2n-1} are listed in Theorem 3.1 in [BS1]. It is not difficult to exclude from this list all domains with automorphism group of dimension less than n^2. This gives that D is CR-equivalent to one of the following domains:

$$\begin{array}{l} \text{(a) } S^{2n-1}, \\ \text{(b) } S^{2n-1} \setminus \{\text{point}\}, \\ \text{(c) } S^{2n-1} \setminus \{z_n = 0\}. \end{array} \tag{3.5}$$

Thus, $\tilde{O}(p)$ is respectively one of the following manifolds:

$$\begin{array}{l} \text{(a) } S^{2n-1}, \\ \text{(b) } \sigma, \\ \text{(c) } \omega. \end{array} \tag{3.6}$$

If $\tilde{O}(p) = S^{2n-1}$, then by Proposition 5.1 of [BS1], the orbit $O(p)$ is CR-equivalent to a lens space as in (i) of (3.3).

Suppose next that $\tilde{O}(p) = \sigma$. The group $\mathrm{Aut}_{CR}(\sigma)$ consists of all maps of the form

$$\begin{aligned} z' &\mapsto \lambda U z' + a, \\ z_n &\mapsto \lambda^2 z_n + 2\lambda \langle U z', a \rangle + |a|^2 + i\alpha, \end{aligned} \tag{3.7}$$

where $U \in U_{n-1}$, $a \in \mathbb{C}^{n-1}$, $\lambda > 0$, $\alpha \in \mathbb{R}$, and, as before, $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{C}^{n-1}$. It then follows that $\mathrm{Aut}_{CR}(\sigma) = CU_{n-1} \ltimes N$, where CU_{n-1} consists of all maps of the form (3.7) with $a = 0$, $\alpha = 0$, and N is the *Heisenberg group* consisting of the maps of the form (3.7) with $U = \mathrm{id}$ and $\lambda = 1$.

Further, description (3.7) implies that $\dim \mathrm{Aut}_{CR}(\sigma) = n^2 + 1$, and therefore $n^2 \leq \dim G \leq n^2 + 1$. If $\dim G = n^2 + 1$, then we have $G = \mathrm{Aut}_{CR}(\sigma)$, and hence Γ lies in the center of $\mathrm{Aut}_{CR}(\sigma)$. Since the center of $\mathrm{Aut}_{CR}(\sigma)$ is trivial, so is Γ. Thus, in this case $O(p)$ is CR-equivalent to the hypersurface σ.

Assume now that $\dim G = n^2$. Using the transitivity of the G-action on σ, it is straightforward to show (for instance, by considering G-vector fields on σ) that $N \subset G$. The centralizer of N in $CU_{n-1} \ltimes N$ consists of all maps of the form

$$\begin{aligned} z' &\mapsto z', \\ z_n &\mapsto z_n + i\alpha, \end{aligned} \tag{3.8}$$

where $\alpha \in \mathbb{R}$. Since Γ acts freely properly discontinuously on σ, it is generated by a single map of the form (3.8) with $\alpha = \alpha_0 \in \mathbb{R}^*$. The hypersurface σ covers the hypersurface

$$\left\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z_n| = \exp\left(\frac{2\pi}{\alpha_0}|z'|^2\right)\right\} \tag{3.9}$$

by means of the map

$$\begin{aligned} z' &\mapsto z', \\ z_n &\mapsto \exp\left(\frac{2\pi}{\alpha_0} z_n\right), \end{aligned} \tag{3.10}$$

and Γ is the group of deck transformations of this map. Hence $O(p)$ is CR-equivalent to hypersurface (3.9). Replacing if necessary z_n by $1/z_n$ we obtain that $O(p)$ is CR-equivalent to the hypersurface δ.

Suppose finally that $\tilde{O}(p) = \omega$. First, we will determine the group $\mathrm{Aut}_{CR}(\omega)$. The general form of a CR-automorphism of $S^{2n-1} \setminus \{z_n = 0\}$ is given by formula (3.1) with $\theta = 1/2$ and the covering map Π by the formula

$$\begin{aligned} z' &\mapsto z', \\ z_n &\mapsto \exp\left(\frac{z_n}{2}\right). \end{aligned}$$

Using (3.4) we then obtain the general form of a CR-automorphism of ω as follows:

$$\begin{aligned} z' &\mapsto \frac{Az' + b}{cz' + d}, \\ z_n &\mapsto z_n - 2\ln(cz' + d) + i\beta, \end{aligned} \tag{3.11}$$

where

$$\begin{pmatrix} A & b \\ c & d \end{pmatrix} \in SU_{n-1,1}, \quad \beta \in \mathbb{R}.$$

In particular, $\mathrm{Aut}_{CR}(\omega)$ is a connected group of dimension n^2 and therefore $G = \mathrm{Aut}_{CR}(\omega)$. Hence Γ is a central subgroup of $\mathrm{Aut}_{CR}(\omega)$. It follows from formula (3.11) that the center of $\mathrm{Aut}_{CR}(\omega)$ consists of all maps of the form (3.8). Hence Γ is generated by a single such map with $\alpha = \alpha_0 \in \mathbb{R}$. If $\alpha_0 = 0$, the orbit $O(p)$ is CR-equivalent to ω. Let $\alpha_0 \neq 0$. The hypersurface ω covers the hypersurface

$$\left\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'|^2 + |z_n|^{\frac{\alpha_0}{2\pi}} = 1,\ z_n \neq 0\right\} \tag{3.12}$$

by means of map (3.10). Since Γ is the group of deck transformations of this map, it follows that $O(p)$ is CR-equivalent to hypersurface (3.12). Replacing if necessary z_n by $1/z_n$, we obtain that $O(p)$ is CR-equivalent to the hypersurface ε_α for some $\alpha > 0$.

The proof is complete. ■

We will now show that the presence of a spherical orbit of a particular kind in M determines the group $G(M)$ as a Lie group. Suppose that for some

$p \in M$ the orbit $O(p)$ is spherical, and let $\mathfrak{m}$ be the manifold from list (3.3) to which $O(p)$ is CR-equivalent. Since $G(M)$ acts properly and effectively on $O(p)$, the CR-equivalence induces an isomorphism between $G(M)$ and a closed connected n^2-dimensional subgroup $R_{\mathfrak{m}}$ of $\mathrm{Aut}_{CR}(\mathfrak{m})$ that acts properly and transitively on $\mathfrak{m}$. A priori, $R_{\mathfrak{m}}$ depends on the choice of a CR-isomorphism between $O(p)$ and $\mathfrak{m}$, but, as we will see shortly, this dependence is insignificant.

We will now prove the following proposition.

Proposition 3.4. ([I2])

(i) $R_{S^{2n-1}}$ is conjugate to U_n in $\mathrm{Aut}_{CR}(S^{2n-1})$, and $R_{\mathcal{L}_m} = U_n/\mathbb{Z}_m$ for $m > 1$;
(ii) R_σ consists of all maps of the form (3.7) with $\lambda = 1$;
(iii) R_δ consists of all maps of the form

$$\begin{aligned} z' &\mapsto Uz' + a, \\ z_n &\mapsto e^{i\beta} \exp\Big(2\langle Uz', a\rangle + |a|^2\Big) z_n, \end{aligned} \tag{3.13}$$

where $U \in U_{n-1}$, $a \in \mathbb{C}^{n-1}$, $\beta \in \mathbb{R}$;
(iv) R_ω consists of all maps of the form (3.11);
(v) R_{ε_α} consists of all maps of the form (3.1) with $\theta = 1/\alpha$.

Proof: Suppose first that $\mathfrak{m} = \mathcal{L}_m$ for some $m \in \mathbb{N}$. Then $O(p)$ is compact and, since I_p is compact as well, it follows that $G(M)$ is compact. Assume first that $m = 1$. In this case $R_{S^{2n-1}}$ is a subgroup of $\mathrm{Aut}_{CR}(S^{2n-1}) = \mathrm{Aut}(\mathbb{B}^n)$. Since $R_{S^{2n-1}}$ is compact, it is conjugate to a subgroup of U_n, which is a maximal compact subgroup in $\mathrm{Aut}(\mathbb{B}^n)$. Since both $R_{S^{2n-1}}$ and U_n are n^2-dimensional, $R_{S^{2n-1}}$ is in fact conjugate to the full group U_n. Suppose now that $m > 1$. It is straightforward to determine the group $\mathrm{Aut}_{CR}(\mathcal{L}_m)$ by lifting CR-automorphisms of $\mathcal{L}_m$ to its universal cover S^{2n-1}. This group is $U_n/\mathbb{Z}_m$ acting on $\mathbb{C}^n \setminus \{0\}/\mathbb{Z}_m$ in the standard way. In particular, $\mathrm{Aut}_{CR}(\mathcal{L}_m)$ is connected and has dimension n^2. Therefore, $R_{\mathcal{L}_m} = U_n/\mathbb{Z}_m$, and we have obtained statement (i) of the proposition.

Assume now that $\mathfrak{m} = \sigma$. Recall that the group $\mathrm{Aut}_{CR}(\sigma)$ consists of all maps of the form (3.7) and has dimension $n^2 + 1$. Since I_p is isomorphic to U_{n-1}, the isotropy subgroup of the origin under the R_σ-action on σ coincides with the group of all maps of the form (3.7) with $\lambda = 1$, $a = 0$, $\alpha = 0$. Furthermore, since R_σ acts transitively on σ, it contains the Heisenberg group N, and statement (ii) of the proposition follows.

Next, the groups $\mathrm{Aut}_{CR}(\delta)$, $\mathrm{Aut}_{CR}(\omega)$, $\mathrm{Aut}_{CR}(\varepsilon_\alpha)$ are n^2-dimensional and connected. Indeed, $\mathrm{Aut}_{CR}(\delta)$ can be determined by considering the universal cover of δ (see the proof of Proposition 3.3) and consists of all maps of the form (3.13). The group $\mathrm{Aut}_{CR}(\omega)$ was found in the proof of Proposition 3.3 and consists of all maps of the form (3.11). The group $\mathrm{Aut}_{CR}(\varepsilon_\alpha)$ can be found by considering the universal cover of ε_α and consists of all maps of the form (3.1) with $\theta = 1/\alpha$. This yields statements (iii), (iv) and (v) of the proposition.

The proof is complete. ■

Next, we classify Levi-flat orbits in the following proposition.

Proposition 3.5. ([I2])*Let M be a connected non-homogeneous hyperbolic manifold of dimension $n \geq 2$ with $d(M) = n^2$. Assume that for a point $p \in M$ its orbit $O(p)$ is Levi-flat. Then $O(p)$ is equivalent to either $\mathbb{B}^{n-1} \times \mathbb{R}$ or $\mathbb{B}^{n-1} \times S^1$ by means of a real-analytic CR-map. The CR-equivalence can be chosen so that it transforms $G(M)|_{O(p)}$ into $R_{\mathbb{B}^{n-1}\times\mathbb{R}}|_{\mathbb{B}^{n-1}\times\mathbb{R}}$, where the group $R_{\mathbb{B}^{n-1}\times\mathbb{R}}$ consists of all maps of the form*

$$(z', z_n) \mapsto (a(z'), z_n + b),$$

in the first case, and into $R_{\mathbb{B}^{n-1}\times S^1}|_{\mathbb{B}^{n-1}\times S^1}$, where the group $R_{\mathbb{B}^{n-1}\times S^1}$ consists of all maps of the form

$$(z', z_n) \mapsto (a(z'), e^{ic} z_n),$$

in the second case, with $a \in Aut(\mathbb{B}^{n-1})$, $b, c \in \mathbb{R}$.

Proof: Recall that the hypersurface $O(p)$ is foliated by complex submanifolds holomorphically equivalent to $\mathbb{B}^{n-1}$ (see (ii) of Proposition 3.2). Denote by $\mathfrak{g}(M)$ the Lie algebra of $G(M)$-vector fields on M. We identify this algebra with the Lie algebra of $G(M)$. Further, we identify every vector field from $\mathfrak{g}(M)$ with its restriction to $O(p)$. For $q \in O(p)$ we consider the leaf $O(p)_q$ of the foliation passing through q and the subspace $\mathfrak{l}_q \subset \mathfrak{g}(M)$ of all vector fields tangent to $O(p)_q$ at q. Since vector fields in $\mathfrak{l}_q$ remain tangent to $O(p)_q$ at each point in $O(p)_q$, the subspace $\mathfrak{l}_q$ is in fact a Lie subalgebra of $\mathfrak{g}(M)$. It follows from the definition of $\mathfrak{l}_q$ that $\dim \mathfrak{l}_q = n^2 - 1$.

Denote by H_q the (possibly non-closed) connected subgroup of $G(M)$ with Lie algebra $\mathfrak{l}_q$. It is straightforward to verify that the group H_q acts on $O(p)_q$ by holomorphic transformations and that $I_q^0 \subset H_q$. If some element $g \in H_q$ acts trivially on $O(p)_q$, then $g \in I_q$. If I_q is isomorphic to U_{n-1}, every element of I_q acts non-trivially on $O(p)_q$ and thus $g = \text{id}$; if I_q is isomorphic to $\mathbb{Z}_2 \times U_{n-1}$ and $g \neq \text{id}$, then $g = g_q$, where g_q denotes the element of I_q corresponding to the non-trivial element in $\mathbb{Z}_2$ (see Case 2 in the proof of Proposition 3.2). Thus, either H_q or $H_q/\mathbb{Z}_2$ acts effectively on $O(p)_q$ (the former case occurs if $g_q \notin H_q$, the latter if $g_q \in H_q$). Since $\dim H_q = n^2 - 1 = d(\mathbb{B}^{n-1})$, we obtain that either H_q or $H_q/\mathbb{Z}_2$ is isomorphic to $\text{Aut}(\mathbb{B}^{n-1})$.

We will now show that in fact $g_q \notin H_q$. Assuming the opposite, we have $I_q \subset H_q$. It then follows that I_q is a maximal compact subgroup of H_q since its image under the projection $H_q \to \text{Aut}(\mathbb{B}^{n-1})$ is a maximal compact subgroup of $\text{Aut}(\mathbb{B}^{n-1})$. However, every maximal compact subgroup of a connected

Lie group is connected whereas I_q is not. Thus, $g_q \notin H_q$, and hence H_q is isomorphic to $\mathrm{Aut}(\mathbb{B}^{n-1})$.

We will now show that $H_{q_1} = H_{q_2}$ for all $q_1, q_2 \in O(p)$. Suppose first that $n \geq 3$. The Lie algebra $\mathfrak{l}_q$ is isomorphic to $\mathfrak{su}_{n-1,1}$. Since the algebra $\mathfrak{su}_{n-1,1}$ does not have codimension 1 subalgebras if $n \geq 3$ (see e.g. [EaI]), $\mathfrak{l}_q$ is the only codimension 1 subalgebra of $\mathfrak{g}(M)$ in this case. This implies that $H_{q_1} = H_{q_2}$ for all $q_1, q_2 \in O(p)$ for $n \geq 3$. Assume now that $n = 2$. By Bochner's linearization theorem (see [Bo]) there exist a local holomorphic change of coordinates F near q on M that identifies an I_q^0-invariant neighborhood U of q with an L_q^0-invariant neighborhood $\mathcal{U}$ of the origin in $T_q(M)$ such that $F(q) = 0$ and $F(gs) = \alpha_q(g)F(s)$ for all $g \in I_q^0$ and $s \in U$. In the proof of Proposition 3.2 (see Case 2) we have seen that L_q^0 fixes every point in the orthogonal complement W_q to $T_q(O(p)_q)$ in $T_q(M)$. Therefore, for all points s in the curve $F^{-1}\Big(\mathcal{U} \cap \Big(W_q \cap T_q(O(p))\Big)\Big)$ we have $I_s^0 = I_q^0$. In particular, $H_q \cap H_s$ contains a subgroup isomorphic to U_1 for such s. Hence $\mathfrak{l}_q \cap \mathfrak{l}_s$ is isomorphic to a subalgebra of $\mathfrak{su}_{1,1}$ of dimension at least 2 that contains a 1-dimensional compact subalgebra. It is straightforward to check, however, that every such subalgebra must coincide with all of $\mathfrak{su}_{1,1}$ (cf. the proof of Lemma 5.3 in Chap. 5), which shows that $H_q = H_s$ for all $s \in F^{-1}\Big(U' \cap \Big(W_q \cap T_q(O(p))\Big)\Big)$ and hence for all s in a neighborhood of q. Since this argument can be applied to any $q \in O(p)$, we obtain that $H_{q_1} = H_{q_2}$ for all $q_1, q_2 \in O(p)$ if $n = 2$ as well. From now on we denote the coinciding groups H_q by H and the coinciding algebras $\mathfrak{l}_q$ by $\mathfrak{l}$. Note that the above argument also shows that H is a normal subgroup of $G(M)$ and $\mathfrak{l}$ is an ideal in $\mathfrak{g}(M)$.

We will now prove that H is closed in $G(M)$. Let U be a neighborhood of 0 in $\mathfrak{g}(M)$ where the exponential map into $G(M)$ is a diffeomorphism, and let $V := \exp(U)$. To prove that H is closed in $G(M)$ it is sufficient to show that for some neighborhood W of $\mathrm{id} \in G(M)$, $W \subset V$, we have $H \cap W = \exp(\mathfrak{l} \cap U) \cap W$. Assuming the opposite we obtain a sequence $\{h_j\}$ of elements of H converging to id in $G(M)$ such that for every j we have $h_j = \exp(a_j)$ with $a_j \in U \setminus \mathfrak{l}$. Fix $q \in O(p)$. There exists a neighborhood $\mathcal{V}$ of q in $O(p)$ that is CR-equivalent to the direct product of $\mathbb{B}^{n-1}$ and a segment in $\mathbb{R}$. For every s in this neighborhood we denote by N_s the complex hypersurface lying in $\mathcal{V}$ that arises from this representation and passes through s. We call such hypersurfaces *local leaves*. Let $q_j := h_j q$. If j is sufficiently large, we have $q_j \in \mathcal{V}$. We will now show that $N_{q_j} \neq N_q$ for large j.

Let $U'' \subset U' \subset U$ be neighborhoods of 0 in $\mathfrak{g}(M)$ such that: (a) $\exp(U'') \cdot \exp(U'') \subset \exp(U')$; (b) $\exp(U'') \cdot \exp(U') \subset \exp(U)$; (c) $U' = -U'$; (d) $I_q \cap \exp(U') \subset \exp(\mathfrak{l} \cap U')$. We also assume that $\mathcal{V}$ is chosen so that $N_q \subset \exp(\mathfrak{l} \cap U'')q$. Suppose that $q_j \in N_q$. Then $q_j = sq$ for some $s \in \exp(\mathfrak{l} \cap U'')$ and hence $t := h_j^{-1}s$ is an element of I_q. For large j we have $h_j^{-1} \in \exp(U'')$. Condition (a) now implies that $t \in \exp(U')$ and hence by (c), (d) we have $t^{-1} \in \exp(\mathfrak{l} \cap U')$. Therefore, by (b) we obtain $h_j \in \exp(\mathfrak{l} \cap U)$ which contradicts our choice of h_j. Thus, for large j the local leaves N_{q_j} are distinct from N_q.

Furthermore, they accumulate to N_q. At the same time we have $N_q \subset O(p)_q$ and $N_{q_j} \subset O(p)_q$ for all j and thus the leaf $O(p)_q$ accumulates to itself. Below we will show that this is in fact impossible thus obtaining a contradiction.

Consider the action of I_q^0 on $O(p)_q$. The orbit $O'(s)$ of every point $s \in O(p)_q$, $s \neq q$, of this action is diffeomorphic to the sphere S^{2n-3} and we have $O(p)_q \setminus O'(s) = V_1(s) \cup V_2(s)$, where $V_1(s)$, $V_2(s)$ are open and disjoint in $O(p)_q$, the point q lies in $V_1(s)$, and $V_2(s)$ is diffeomorphic to a spherical shell in $\mathbb{C}^{n-1}$. Since I_q^0 is compact, there exist neighborhoods $\mathcal{V}$, $\mathcal{V}'$ of q in $O(p)$, $\mathcal{V}' \subset \mathcal{V}$, such that $\mathcal{V}$ is represented as a union of local leaves, and $I_q^0 \mathcal{V}' \subset \mathcal{V}$. Since $O(p)_q$ accumulates to itself near q, there exists $s_0 \in O(p)_q \cap \mathcal{V}'$, $s_0 \notin N_q$. Clearly, $O'(s_0) \subset \mathcal{V}$. Since the I_q^0-vector fields in $\mathfrak{g}(M)$ are tangent to the local leaf $N_{s_0} \subset O(p)_q$ at s_0 and $\mathcal{V}$ is partitioned into non-intersecting local leaves, the orbit $O'(s_0)$ lies in N_{s_0}. Then we have $N_{s_0} \setminus O'(s_0) = W_1(s_0) \cup W_2(s_0)$, where $W_1(s_0)$ is diffeomorphic to $\mathbb{B}^{n-1}$. Since $q \notin N_{s_0}$, we have $V_2(s_0) = W_1(s_0)$, which is impossible since in this case $V_2(s_0)$ is not diffeomorphic to a spherical shell. This contradiction shows that H is closed in $G(M)$.

Thus, since the action of $G(M)$ is proper on M, the action of H on $O(p)$ is proper as well, and the orbits of this action are the leaves of the foliation. Since all isotropy subgroups for the action of H on $O(p)$ are conjugate to each other, the orbit space $O(p)/H$ is homeomorphic to either $\mathbb{R}$ or S^1 (see [AA]). By [Pa] (see also [Alek]) there is a complete H-invariant Riemannian metric on $O(p)$. Fix $p_0 \in O(p)$, let $F : O(p)_{p_0} \to \mathbb{B}^{n-1}$ be a biholomorphism, and consider a normal geodesic γ_0 emanating from p_0 (see [AA]). We will think of γ_0 as a parametrized curve, with $\gamma_0(0) = p_0$.

Suppose first that $O(p)/H$ is homeomorphic to $\mathbb{R}$. We will now construct a CR-isomorphism $F_1 : O(p) \to \mathbb{B}^{n-1} \times \mathbb{R}$ using the properties of normal geodesics listed in Proposition 4.1 of [AA]. In this case γ_0 is diffeomorphic to $\mathbb{R}$. For $q \in O(p)$ consider $O(p)_q$, and let r be the (unique) point where γ_0 intersects $O(p)_q$. Let $h \in H$ be such that $q = hr$ and t_q be the value of the parameter on γ such that $\gamma(t_q) = r$. Then we set $F_1(q) := (F(hp_0), t_q)$. By construction, F_1 is a real-analytic CR-map.

Suppose now that $O(p)/H$ is homeomorphic to S^1. We will construct a CR-isomorphism $F_2 : O(p) \to \mathbb{B}^{n-1} \times S^1$ using the properties of normal geodesics from Proposition 4.1 and Theorems 5.1, 6.1 of [AA]. Following [AA], consider the Weyl group $W(\gamma_0)$ of γ_0. This group can be identified with a subgroup of the group $N_H(I_q^0)/I_q^0$, where q is any point in γ_0 and $N_H(I_q^0)$ is the normalizer of I_q^0 in H. Since H is isomorphic to $\mathrm{Aut}(\mathbb{B}^{n-1})$ and I_q^0 upon this identification is conjugate to $U_{n-1} \subset \mathrm{Aut}(\mathbb{B}^{n-1})$, we see that $N_H(I_q^0) = I_q^0$ and thus $W(\gamma_0)$ is trivial. This implies that γ_0 intersects every $O(p)_q$ at exactly one point and is diffeomorphic to S^1 as a subset of $O(p)$. For $q \in O(p)$ consider $O(p)_q$, and let r be the point where γ_0 intersects $O(p)_q$. Let $h \in H$ be such that $q = hr$ and t_q be the least value of the parameter on γ_0 such that $\gamma_0(t_q) = r$. Then we set $F_2(q) := (F(hp_0), e^{2\pi i t_q/T})$, where T is the least positive value of the parameter for which $\gamma_0(T) = p_0$. The CR-map F_2 is real-analytic.

It follows from the construction of the maps F_1, F_2 that the elements of $H' := F_j \circ H|_{O(p)} \circ F_j^{-1}$ for the relevant j have the form

$$(z', u) \mapsto (a(z'), u),$$

where $a \in \mathrm{Aut}(\mathbb{B}^{n-1})$ and (z', u) is a point in either $\mathbb{B}^{n-1} \times \mathbb{R}$, or $\mathbb{B}^{n-1} \times S^1$. We will now find the general form of the elements of the group $G' := F_j \circ G(M)|_{O(p)} \circ F_j^{-1}$.

Every CR-isomorphism of either $\mathbb{B}^{n-1} \times \mathbb{R}$, or $\mathbb{B}^{n-1} \times S^1$ has the form

$$(z', u) \mapsto (a_u(z'), \mu(u)), \tag{3.14}$$

where $a_u \in \mathrm{Aut}(\mathbb{B}^{n-1})$ for every u, and μ is a diffeomorphism of either $\mathbb{R}$ or S^1, respectively. Since H is normal in $G(M)$, H' is normal in G'. Fix an element of G' and let $\{a_u\}$ be the corresponding family of automorphisms of $\mathbb{B}^{n-1}$, as in (3.14). Then we have $a_{u_1} a a_{u_1}^{-1} = a_{u_2} a a_{u_2}^{-1}$ for all $a \in \mathrm{Aut}(\mathbb{B}^{n-1})$ and all u_1, u_2. Therefore, $a_{u_1}^{-1} a_{u_2}$ lies in the center of $\mathrm{Aut}(\mathbb{B}^{n-1})$, which is trivial. Hence we obtain that $a_{u_1} = a_{u_2}$ for all u_1, u_2. This shows that every element of G' is a composition of an element of H' and an element of a one-parameter family of real-analytic automorphism of the form

$$(z', u) \mapsto (z', \mu(u)).$$

Since for every $q \in O(p)$ there can exist at most one element of $G(M)$ outside H that fixes q (namely, the transformation g_q), the corresponding one-parameter family $\{\mu_\tau\}_{\tau \in \mathbb{R}}$ has no fixed points in either $\mathbb{R}$ or S^1, respectively. If $O(p)$ is equivalent to $\mathbb{B}^{n-1} \times \mathbb{R}$, then under the diffeomorphism of $\mathbb{R}$ inverse to the map $x \mapsto \mu_x(0)$, the family $\{\mu_\tau\}_{\tau \in \mathbb{R}}$ transforms into the family

$$(z', u) \mapsto (z', u + \tau).$$

If $O(p)$ is equivalent to $\mathbb{B}^{n-1} \times S^1$, then the diffeomorphism of S^1

$$e^{i\theta} \mapsto \exp\left(\frac{2\pi i}{c} \int_0^\theta \frac{1}{V(t)}\, dt\right),$$

with $V(t) := d\mu_\tau/d\tau|_{\tau=0, u=e^{it}}$ and $c = \int_0^{2\pi} dt/V(t)$, $0 \le t, \theta \le 2\pi$, transforms the family $\{\mu_\tau\}_{\tau \in \mathbb{R}}$ into the family

$$(z', u) \mapsto (z', e^{\frac{2\pi i}{c}\tau} u).$$

The diffeomorphisms of $\mathbb{R}$ and S^1 constructed above are real-analytic.

The proof is complete. ■

The hypersurfaces determined in Propositions 3.3 and 3.5 will be called the *models* for real hypersurface orbits. For a Levi-flat orbit we will always assume that a CR-equivalence between the orbit and the corresponding model is chosen to ensure that $G(M)|_{O(p)}$ is transformed into either $R_{\mathbb{B}^{n-1}\times\mathbb{R}}|_{\mathbb{B}^{n-1}\times\mathbb{R}}$ or $R_{\mathbb{B}^{n-1}\times S^1}|_{\mathbb{B}^{n-1}\times S^1}$.

It is straightforward to see that all models are pairwise CR non-equivalent. For all models, except the family $\{\varepsilon_\alpha\}_{\alpha>0}$, this follows from topological considerations and comparison of the automorphism groups. Furthermore, any CR-isomorphism between ε_α and ε_β can be lifted to a CR-automorphism of ω, the universal cover of each of ε_α and ε_β, and it is straightforward to observe that in order for the lifted map to be well-defined and one-to-one, we must have $\beta = \alpha$.

3.4 Proof of Theorem 3.1

In this section we prove Theorem 3.1 by studying how real and complex hypersurface orbits can be glued together to form hyperbolic manifolds with n^2-dimensional automorphism groups.

First of all, suppose that for some $p \in M$ the model for $O(p)$ is S^{2n-1}. It then follows from Proposition 3.4 that M admits an effective action of U_n by holomorphic transformations. All connected n-dimensional manifolds that admit such actions were listed in [IKru1] (see Sect. 6.2). The only hyperbolic manifolds with n^2-dimensional automorphism group on the list are quotients of spherical shells $S_r/\mathbb{Z}_l$ (see (2.1)), for $0 \le r < 1$ and $l = |nk+1|$ with $k \in \mathbb{Z}$ (note that for such l the groups U_n and $U_n/\mathbb{Z}_l$ are isomorphic – see (6.7)). However, for $l > 1$ no orbit of the action of $G(S_r/\mathbb{Z}_l) = U_n/\mathbb{Z}_l$ on $S_r/\mathbb{Z}_l$ is diffeomorphic to S^{2n-1}, which gives that M is in fact equivalent to S_r. From now on we assume that S^{2n-1} is not the model for any real hypersurface orbit in M.

Next, we observe that for every model $\mathfrak{m}$ the group $R_\mathfrak{m}$ acts by holomorphic transformations with real hypersurface orbits on a certain manifold $M_\mathfrak{m}$ containing $\mathfrak{m}$, such that every orbit of the $R_\mathfrak{m}$-action on $M_\mathfrak{m}$ is equivalent to $\mathfrak{m}$ by means of a holomorphic automorphism of $M_\mathfrak{m}$ of a simple form. We list the manifolds $M_\mathfrak{m}$ and corresponding automorphisms below.

(a) $M_{\mathcal{L}_m} = \mathbb{C}^n \setminus \{0\}/\mathbb{Z}_m$ for $m > 1$; every two $R_{\mathcal{L}_m}$-orbits in $\mathbb{C}^n \setminus \{0\}/\mathbb{Z}_m$ are equivalent by means of a map of the form

$$[z] \mapsto [rz], \tag{3.15}$$

where $r > 0$ and $[z] \in \mathbb{C}^n \setminus \{0\}/\mathbb{Z}_m$ denotes the equivalence class of $z \in \mathbb{C}^n \setminus \{0\}$.

(b) $M_\sigma = \mathbb{C}^n$; every two R_σ-orbits in $\mathbb{C}^n$ are equivalent by means of a real translation in z_n.

(c) $M_\delta = \mathbb{C}^n$; every two R_δ-orbits in $\mathbb{C}^n$ are equivalent by means of a real dilation in z_n.

(d) $M_\omega = \mathcal{C} := \{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'| < 1\}$; every two R_ω-orbits in $\mathcal{C}$ are equivalent by means of a real translation in z_n.

(e) $M_{\varepsilon_\alpha} = \mathcal{C}_0 := \mathcal{C} \setminus \{z_n = 0\}$; every two R_{ε_α}-orbits in $\mathcal{C}_0$ are equivalent by means of a real dilation in z_n.

(f) $M_{\mathbb{B}^{n-1}\times\mathbb{R}} = \mathcal{C}$; every two $R_{\mathbb{B}^{n-1}\times\mathbb{R}}$-orbits in $\mathcal{C}$ are equivalent by means of an imaginary translation in z_n.

(g) $M_{\mathbb{B}^{n-1}\times S^1} = \mathcal{C}_0$; every two $R_{\mathbb{B}^{n-1}\times S^1}$-models in $\mathcal{C}_0$ are equivalent by means of a real dilation in z_n.

We now assume that every orbit in M is a real hypersurface. Our orbit gluing procedure in this case comprises the following steps.

(I). Start with a real hypersurface orbit $O(p)$ with model $\mathfrak{m}$ and consider a real-analytic CR-isomorphism $f : O(p) \to \mathfrak{m}$. The map f satisfies

$$f(gq) = \varphi(g)f(q), \tag{3.16}$$

for all $g \in G(M)$ and $q \in O(p)$, where $\varphi : G(M) \to R_\mathfrak{m}$ is a Lie group isomorphism.

(II). Observe that f can be extended to a biholomorphic map from a $G(M)$-invariant connected neighborhood of $O(p)$ in M onto an $R_\mathfrak{m}$-invariant neighborhood of $\mathfrak{m}$ in the corresponding manifold $M_\mathfrak{m}$. If $G(M)$ is compact (in which case $\mathfrak{m}$ is a lens space $\mathcal{L}_m$ with $m > 1$), then every neighborhood of $O(p)$ contains a $G(M)$-invariant neighborhood. In this case, we extend f biholomorphically to some neighborhood of $O(p)$ (this can be done due to the real-analyticity of f) and choose a $G(M)$-invariant neighborhood in it.

We now assume that $G(M)$ is non-compact. It will be more convenient for us to extend the inverse map $\mathfrak{F} := f^{-1}$. First of all, extend $\mathfrak{F}$ to some neighborhood U of $\mathfrak{m}$ in $M_\mathfrak{m}$ to a biholomorphic map onto a neighborhood W of $O(p)$ in M. One can see from the explicit form of the $R_\mathfrak{m}$-action on the manifold $M_\mathfrak{m}$ that a neighborhood U can be chosen to satisfy the following condition that we call Condition ($*$): for every two points $s_1, s_2 \in U$ and every element $h \in R_\mathfrak{m}$ such that $hs_1 = s_2$ there exists a curve $\gamma \subset U$ joining s_1 with a point in $\mathfrak{m}$ for which $h\gamma \subset U$ (clearly, $h\gamma$ is a curve joining s_2 with a point in $\mathfrak{m}$).

To extend $\mathfrak{F}$ to a $R_\mathfrak{m}$-invariant neighborhood of $\mathfrak{m}$, fix $s \in U$ and $s_0 \in \mathcal{O}(s)$, where $\mathcal{O}(q)$ denotes the $R_\mathfrak{m}$-orbit of a point $q \in M_\mathfrak{m}$. Choose $h_0 \in R_\mathfrak{m}$ such that $s_0 = h_0 s$ and define $\mathfrak{F}(s_0) := \varphi^{-1}(h_0)\mathfrak{F}(s)$. We will now show that $\mathfrak{F}$ is well-defined. Suppose that for some $s_1, s_2 \in U$ and $h_1, h_2 \in R_\mathfrak{m}$ we have $s_0 = h_1 s_1 = h_2 s_2$. To show that $\varphi^{-1}(h_1)\mathfrak{F}(s_1) = \varphi^{-1}(h_2)\mathfrak{F}(s_2)$ we set $h := h_2^{-1}h_1$ and, according to Condition ($*$), find a curve $\gamma \subset U$ that joins s_1 with a point in $\mathfrak{m}$ and such that $h\gamma \subset U$. Clearly, for all $q \in \mathfrak{m}$ we have

$$\mathfrak{F}(hq) = \varphi^{-1}(h)\mathfrak{F}(q) \tag{3.17}$$

(see (3.16)). Consider the set $h^{-1}U \cap U$ and let U_h be its connected component containing $\mathfrak{m}$. For $q \in U_h$ identity (3.17) holds. It now follows from the existence of a curve γ as above that $s_1 \in U_h$. Thus, (3.17) holds for $q = s_1$, and we have shown that $\mathfrak{F}$ is well-defined at s_0. The same argument gives that for $s_0 \in U$ our definition agrees with the original value $\mathfrak{F}(s_0)$. Thus, we have extended $\mathfrak{F}$ to $U' := \cup_{s \in U} \mathcal{O}(s)$. The extended map is locally biholomorphic, satisfies (3.17), and maps U' onto a $G(M)$-invariant neighborhood W' of $O(p)$ in M. We will now show that the extended map is 1-to-1 on U'.

Suppose that for some $s_0, s_0' \in U'$, $s_0 \neq s_0'$, we have $\mathfrak{F}(s_0) = \mathfrak{F}(s_0')$. This can only occur if s_0 and s_0' lie in the same $R_{\mathfrak{m}}$-orbit, and therefore there exists a point $s \in U$ and elements $h, h' \in R_{\mathfrak{m}}$ such that $s_0 = hs$, $s_0' = h's$. Then we have $h'^{-1}h \notin J_s$ and $\varphi^{-1}(h'^{-1}h) \in I_{\mathfrak{F}(s)}$, where J_s denotes the isotropy subgroup of s under the $R_{\mathfrak{m}}$-action. At the same time, we have $\varphi^{-1}(J_s) \subset I_{\mathfrak{F}(s)}$. It follows from the explicit forms of the models and the corresponding groups that $I_{\mathfrak{F}(s)}$ and J_s are isomorphic to U_{n-1}. Hence we have $\varphi^{-1}(J_s) = I_{\mathfrak{F}(s)}$ which contradicts the fact that $h'^{-1}h \notin J_s$.

Thus we have shown that f can be extended to a biholomorphic map satisfying (3.16) between a $G(M)$-invariant neighborhood of $O(p)$ in M and a $R_{\mathfrak{m}}$-invariant neighborhood of $\mathfrak{m}$ in $M_{\mathfrak{m}}$.

(III). Consider a maximal $G(M)$-invariant domain $D \subset M$ from which there exists a biholomorphic map f onto an $R_{\mathfrak{m}}$-invariant domain in $M_{\mathfrak{m}}$ satisfying (3.16) for all $g \in G(M)$ and $q \in D$. The existence of such a domain is guaranteed by the previous step. Assume that $D \neq M$ and consider $x \in \partial D$. Let $\mathfrak{m}_1$ be the model for $O(x)$ and let $f_1 : O(x) \to \mathfrak{m}_1$ be a real-analytic CR-isomorphism satisfying (3.16) for all $g \in G(M)$, $q \in O(x)$ and some Lie group isomorphism $\varphi_1 : G(M) \to R_{\mathfrak{m}_1}$ in place of φ. As in (III), extend f_1 to a biholomorphic map from a connected $G(M)$-invariant neighborhood V of $O(x)$ onto an $R_{\mathfrak{m}_1}$-invariant neighborhood of $\mathfrak{m}_1$ in $M_{\mathfrak{m}_1}$. The extended map satisfies (3.16) for all $g \in G(M)$, $q \in V$ and φ_1 in place of φ. Consider $s \in V \cap D$. The maps f and f_1 take $O(s)$ onto an $R_{\mathfrak{m}}$-orbit $\mathfrak{m}'$ in $M_{\mathfrak{m}}$ and an $R_{\mathfrak{m}_1}$-orbit $\mathfrak{m}_1'$ in $M_{\mathfrak{m}_1}$, respectively. Then $F := f_1 \circ f^{-1}$ maps $\mathfrak{m}'$ onto $\mathfrak{m}_1'$. Since the hypersurface $\mathfrak{m}'$ is CR-equivalent to $\mathfrak{m}$, the hypersurface $\mathfrak{m}_1'$ is CR-equivalent to $\mathfrak{m}_1$ and all models are pairwise CR non-equivalent, we obtain $\mathfrak{m}_1 = \mathfrak{m}$. Clearly, F is a composition of an element of $\mathrm{Aut}_{CR}(\mathfrak{m}')$ and a map that takes $\mathfrak{m}'$ into $\mathfrak{m}_1'$, as listed in (a)–(g) above. We now need to show that F extends to a holomorphic automorphism of $M_{\mathfrak{m}}$. If $\mathfrak{m}$ is spherical, this follows from the fact that every element of $\mathrm{Aut}_{CR}(\mathfrak{m}')$ extends to an automorphism of $M_{\mathfrak{m}}$; if $\mathfrak{m}$ is Levi-flat, an additional argument will be required.

(IV). The neighborhood V at step (III) can be chosen so that we have $V = V_1 \cup V_2 \cup O(x)$, where V_j are open connected non-intersecting sets. For spherical $\mathfrak{m}$ the existence of such V follows, for example, from the strong pseudoconvexity of $\mathfrak{m}$, for Levi-flat $\mathfrak{m}$ it follows from the explicit form of the models: indeed, each of $\mathbb{B}^{n-1}\times\mathbb{R}$, $\mathbb{B}^{n-1}\times S^1$ splits $\mathbb{B}^{n-1}\times\mathbb{C}$. Next, if V is sufficiently small, then each V_j either is a subset of D or is disjoint from it. Suppose first that there is only one j for which $V_j\subset D$. In this case $V\cap D$ is connected and $V\setminus(D\cup O(x))\neq\emptyset$. Setting now

$$\tilde{f} := \begin{cases} f & \text{on } D,\\ F^{-1}\circ f_1 & \text{on } V,\end{cases} \tag{3.18}$$

we obtain a biholomorphic extension of f to $D\cup V$. By construction, $\tilde{f}$ satisfies (3.16) for $g\in G(M)$ and $q\in D\cup V$. Since $D\cup V$ is strictly larger than D, we obtain a contradiction with the maximality of D. Thus, in this case $D=M$, and hence M is holomorphically equivalent to an $R_{\mathfrak{m}}$-invariant domain in $M_{\mathfrak{m}}$ (all such domains will explicitly appear below).

Suppose now that $V_j\subset D$ for $j=1,2$. Applying formula (3.18) to suitable f_1 and F, we can extend $f|_{V_1}$ and $f|_{V_2}$ to biholomorphic maps $\hat{f}_1$, $\hat{f}_2$, respectively, from a neighborhood of $O(x)$ into $M_{\mathfrak{m}}$; each of these maps satisfies (3.16). Let $\hat{\mathfrak{m}}_j := \hat{f}_j(O(x))$, $j=1,2$. Then $\partial D=\hat{\mathfrak{m}}_1\cup\hat{\mathfrak{m}}_2$, $\hat{\mathfrak{m}}_1\neq\hat{\mathfrak{m}}_2$, and $M\setminus O(x)$ is holomorphically equivalent to D. The map $\hat{F}:=\hat{f}_2\circ\hat{f}_1^{-1}$ is a CR-isomorphism from $\hat{\mathfrak{m}}_1$ onto $\hat{\mathfrak{m}}_2$, and M is holomorphically equivalent to the manifold $M_{\hat{F}}$ obtained from $\overline{D}$ by identifying $\hat{\mathfrak{m}}_1$ with $\hat{\mathfrak{m}}_2$ by means of $\hat{F}$. Since the action of $R_{\mathfrak{m}}$ on $M\setminus O(x)$ extends to an action on M, the map $\hat{F}$ is $R_{\mathfrak{m}}$-equivariant. In each of cases (a)–(g) this will imply that $\hat{F}$ extends to a holomorphic automorphism of $M_{\mathfrak{m}}$ of a simple form similar to the corresponding form appearing on list (a)–(g). Let Γ denote the group of automorphisms of $M_{\mathfrak{m}}$ generated by $\hat{F}$. It will follow from the explicit forms of $\hat{F}$ and $M_{\mathfrak{m}}$ in each of cases (a)–(g) that Γ acts freely properly discontinuously on $M_{\mathfrak{m}}$ and that $M_{\mathfrak{m}}$ covers M, with Γ being the group of deck transformations of the covering map. Observe, however, that for every $\mathfrak{m}$ the manifold $M_{\mathfrak{m}}$ is not hyperbolic. This contradiction shows that exactly one of V_j, $j=1,2$, is a subset of D, and hence M is holomorphically equivalent to an $R_{\mathfrak{m}}$-invariant domain in $M_{\mathfrak{m}}$.

We now apply our general orbit gluing procedure in each of cases (a)–(g). In case (a) any $R_{\mathcal{L}_m}$-invariant domain has the form

$$\{z\in\mathbb{C}^n : r<|z|<R\}/\mathbb{Z}_m, \tag{3.19}$$

where $0\le r<R\le\infty$ (for a hyperbolic domain we necessarily have $R<\infty$). If at step (IV) we have $V_j\subset D$ for $j=1,2$, then for the domain D we have $r>0$, and $\hat{F}$ has the form (3.15) with $r\in\mathbb{C}^*$, $|r|\neq 1$ (hence $M_{\hat{F}}$ is a Hopf manifold – see Sect. 6.2). In this case M is holomorphically equivalent to $S_{r/R}/\mathbb{Z}_m$ (see (2.1)).

In case (b) any R_σ-invariant domain has the form

$$\left\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : r + |z'|^2 < \operatorname{Re} z_n < R + |z'|^2\right\}, \tag{3.20}$$

where $-\infty \le r < R \le \infty$. If at step (IV) we have $V_j \subset D$ for $j = 1, 2$, then for the domain D we have $r > -\infty$, $R < \infty$, and $\hat{F}$ is a translation in z_n. In this case M is holomorphically equivalent either to the domain $\mathfrak{C}$ (see (3.2)) or (for $R = \infty$) to $\mathbb{B}^n$; the latter is clearly impossible.

In case (c) any R_δ-invariant domain is given by

$$\left\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : r \exp\left(|z'|^2\right) < |z_n| < R \exp\left(|z'|^2\right)\right\}, \tag{3.21}$$

for $0 \le r < R \le \infty$. If at step (IV) we have $V_j \subset D$ for $j = 1, 2$, then for the domain D we have $r > 0$, $R < \infty$, and $\hat{F}$ is a dilation in z_n. In this case M is holomorphically equivalent either to $\mathfrak{A}_{r/R,\,1}$ or (for $R = \infty$) to $\mathfrak{A}_{0,-1}$ (see (3.2)).

In case (d) any R_ω-invariant domain is of the form

$$\begin{aligned} \Big\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'| < 1,& \\ r(1 - |z'|^2) < \exp\left(\operatorname{Re} z_n\right) < R(1 - |z'|^2)\Big\},& \end{aligned} \tag{3.22}$$

for $0 \le r < R \le \infty$. If at step (IV) we have $V_j \subset D$ for $j = 1, 2$, then for the domain D we have $r > 0$, $R < \infty$, and $\hat{F}$ is a translation in z_n. In this case M is holomorphically equivalent either to $\mathfrak{B}_{r/R,\,1}$ or (for $R = \infty$) to $\mathfrak{B}_{0,-1}$ (see (3.2)).

In case (e) any R_{ε_α}-invariant domain is given by

$$\begin{aligned} \mathcal{E}^R_{r,\alpha} := \Big\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'| < 1,& \\ r(1 - |z'|^2)^{1/\alpha} < |z_n| < R(1 - |z'|^2)^{1/\alpha}\Big\},& \end{aligned} \tag{3.23}$$

for $0 \le r < R \le \infty$. If at step (IV) we have $V_j \subset D$ for $j = 1, 2$, then for the domain D we have $r > 0$, $R < \infty$, and $\hat{F}$ is a dilation in z_n. In this case M is holomorphically equivalent either to $\mathcal{E}_{r/R,1/\alpha}$ or (for $R = \infty$) to $\mathcal{E}_{0,-1/\alpha}$ (see (3.2)).

Consider now case (f). We need to show that the map F that arises at step (III) extends to a holomorphic automorphism of $\mathcal{C}$. The $R_{\mathbb{B}^{n-1}\times\mathbb{R}}$-orbit of every point in $\mathcal{C}$ is of the form

$$b_r := \left\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'| < 1, \operatorname{Im} z_n = r\right\},$$

for $r \in \mathbb{R}$. Since F maps b_{r_1} onto b_{r_2}, for some $r_1, r_2 \in \mathbb{R}$, we have $F = \nu \circ g$, where ν is an imaginary translation in z_n, and $g \in \operatorname{Aut}_{CR}(b_{r_1})$. The maps f and f_1 transform the group $G(M)|_{O(x)}$ into the groups $R_{\mathbb{B}^{n-1}\times\mathbb{R}}|_{b_{r_1}}$ and $R_{\mathbb{B}^{n-1}\times\mathbb{R}}|_{b_{r_2}}$, respectively, and therefore g lies in the normalizer of $R_{\mathbb{B}^{n-1}\times\mathbb{R}}|_{b_{r_1}}$ in $\operatorname{Aut}_{CR}(b_{r_1})$. Considering g in the general form (3.14), we obtain $a_{u_1} a a_{u_1}^{-1} = a_{u_2} a a_{u_2}^{-1}$ for all $a \in \operatorname{Aut}(\mathbb{B}^{n-1})$ and all u_1, u_2. Therefore, $a_{u_1}^{-1} a_{u_2}$ lies in the center of $\mathrm{Aut}(\mathbb{B}^{n-1})$, which is trivial. Hence we obtain that $a_{u_1} = a_{u_2}$ for all u_1, u_2. In addition, there exists $d \in \mathbb{R}^*$ such that $\mu^{-1}(u) + b \equiv \mu^{-1}(u + db)$ for all $b \in \mathbb{R}$. Differentiating this identity with respect to b at $b = 0$ we see that $\mu^{-1}(u) = u/d + t$ for some $t \in \mathbb{R}$. Therefore, F extends to a holomorphic automorphism of $\mathcal{C}$ as the following map

$$\begin{aligned} z' &\mapsto a(z'),\\ z_n &\mapsto d(z_n - t) + i\sigma, \end{aligned} \tag{3.24}$$

where $a \in \mathrm{Aut}(\mathbb{B}^{n-1})$, $\sigma \in \mathbb{R}$.

Any $R_{\mathbb{B}^{n-1}\times\mathbb{R}}$-invariant domain is given by

$$\left\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'| < 1,\ r < \mathrm{Im}\, z_n < R\right\},$$

for $-\infty \le r < R \le \infty$. If at step (IV) we have $V_j \subset D$ for $j = 1, 2$, then for the domain D we have $r > -\infty$, $R < \infty$, and the argument applied above to the map F shows that $\hat{F}$ has the form (3.24). Further, using the fact that $\hat{F}$ is $R_{\mathbb{B}^{n-1}\times\mathbb{R}}$-equivariant, we obtain that $\hat{F}$ is a translation in z_n. In this case M is holomorphically equivalent to $\mathbb{B}^{n-1} \times \Delta$, which is impossible. This shows that M in fact does not contain orbits with model $\mathbb{B}^{n-1} \times \mathbb{R}$.

Finally, consider case (g). We need to show that the map F from step (III) extends to a holomorphic automorphism of $\mathcal{C}_0$. The $R_{\mathbb{B}^{n-1}\times S^1}$-orbit of a point in $\mathcal{C}_0$ has the form

$$c_r := \left\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'| < 1,\ |z_n| = r\right\},$$

for $r > 0$. Since F maps c_{r_1} onto c_{r_2}, for some $r_1, r_2 > 0$, we have $F = \nu \circ g$, where ν is a real dilation in z_n, and $g \in \mathrm{Aut}_{CR}(c_{r_1})$. Analogously to the previous case, the maps f and f_1 transform the group $G(M)|_{O(x)}$ into the groups $R_{\mathbb{B}^{n-1}\times S^1}|_{c_{r_1}}$ and $R_{\mathbb{B}^{n-1}\times S^1}|_{c_{r_2}}$, respectively, hence the element g lies in the normalizer of $R_{\mathbb{B}^{n-1}\times S^1}$ in $\mathrm{Aut}_{CR}(c_{r_1})$. As before, we take g in the general form (3.14) and obtain that $a_{u_1} = a_{u_2}$ for all u_1, u_2. Furthermore, we have $e^{ic}\mu^{-1}(u) \equiv \mu^{-1}(e^{\pm ic}u)$. Differentiating this identity with respect to c at $c = 0$ we see that $\mu^{-1}(u) = e^{it}u^{\pm 1}$, for some $t \in \mathbb{R}$. Therefore, F extends to a holomorphic automorphism of $\mathcal{C}_0$ of the form

$$\begin{aligned} z' &\mapsto a(z'),\\ z_n &\mapsto \rho z_n^{\pm 1} e^{\mp it}, \end{aligned} \tag{3.25}$$

where $\rho \in \mathbb{R}$.

Any $R_{\mathbb{B}^{n-1}\times S^1}$-invariant domain in $\mathcal{C}_0$ is given by

$$C_r^R := \left\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'| < 1,\ r < |z_n| < R\right\},$$

for $0 \le r < R \le \infty$. If at step (IV) we have $V_j \subset D$ for $j = 1, 2$, then for the domain D we have $r > 0$, $R < \infty$, and the argument applied above to the map F shows that $\hat{F}$ has the form (3.25). Further, using the fact that

$\hat{F}$ is $R_{\mathbb{B}^{n-1}\times S^1}$-equivariant, we obtain that $\hat{F}$ is a dilation in z_n. In this case M is holomorphically equivalent either to $\mathcal{E}_{r/R,0}$ or (for $R = \infty$) to $\mathcal{E}_{0,0}$ (see (3.2)). This completes the case when M does not contain complex hypersurface orbits.

We now assume that a complex hypersurface orbit is present in M. Recall that there are at most two such orbits and that the quotient of $G(M)$ by a normal subgroup isomorphic to U_1 is isomorphic to $\mathrm{Aut}(\mathbb{B}^{n-1})$ (see (iii) of Proposition 3.2). It now follows from Propositions 3.4 and 3.5 that the model for every real hypersurface orbit in M is either ε_α for some $\alpha > 0$, or $\mathbb{B}^{n-1}\times S^1$. Let M' be the manifold obtained from M by removing all complex hypersurface orbits. It then follows from the above considerations that M' is holomorphically equivalent to either $\mathcal{E}^R_{r,\alpha}$ or to C^R_r for some $0 \le r < R \le \infty$.

Suppose first that M' is equivalent to $\mathcal{E}^R_{r,\alpha}$ and let $f : M' \to \mathcal{E}^R_{r,\alpha}$ be a biholomorphic map satisfying (3.16) for all $g \in G(M)$, $q \in M'$ and some isomorphism $\varphi : G(M) \to R_{\varepsilon_\alpha}$. The group R_{ε_α} in fact acts on all of $\mathcal{C}$, and the orbit of any point in $\mathcal{C}$ with $z_n = 0$ is the complex hypersurface

$$c_0 := \left\{(z', z_n) \in \mathbb{C}^{n-1}\times\mathbb{C} : |z'| < 1,\ z_n = 0\right\}. \tag{3.26}$$

For a point $s \in \mathcal{C}$ denote by J_s the isotropy subgroup of s under the action of R_{ε_α}. If $s_0 \in c_0$ and $s_0 = (z_0', 0)$, then J_{s_0} is isomorphic to $U_1 \times U_{n-1}$ and consists of all maps of the form (3.1) with $\theta = 1/\alpha$ for which the transformations in the z'-variables form the isotropy subgroup of the point z_0' in $\mathrm{Aut}(\mathbb{B}^{n-1})$.

Fix $s_0 = (z_0', 0) \in c_0$ and let

$$N_{s_0} := \left\{s \in \mathcal{E}^R_{r,\alpha} : J_s \subset J_{s_0}\right\}. \tag{3.27}$$

We have

$$N_{s_0} = \Big\{(z', z_n) \in \mathbb{C}^{n-1}\times\mathbb{C} : z' = z_0',\ r(1 - |z_0'|^2)^{1/\alpha} < |z_n| < R(1 - |z_0'|^2)^{1/\alpha}\Big\}. \tag{3.28}$$

Thus, N_{s_0} is either an annulus (possibly, with infinite outer radius) or a punctured disk. In particular, N_{s_0} is a complex curve in $\mathcal{C}_0$.

Since J_{s_0} is a maximal compact subgroup of R_{ε_α}, the group $\varphi^{-1}(J_{s_0})$ is a maximal compact subgroup of $G(M)$. Let O be a complex hypersurface orbit in M. For $q \in O$ the isotropy subgroup I_q is isomorphic to $U_1 \times U_{n-1}$ and therefore is a maximal compact subgroup of $G(M)$ as well. Thus, $\varphi^{-1}(J_{s_0})$ is conjugate to I_q for every $q \in O$ and hence there exists $q_0 \in O$ such that $\varphi^{-1}(J_{s_0}) = I_{q_0}$. Since the isotropy subgroups in R_{ε_α} of distinct points in c_0 do not coincide, such a point q_0 is unique.

Let

$$K_{q_0} := \{q \in M' : I_q \subset I_{q_0}\}. \tag{3.29}$$

Clearly, $K_{q_0} = f^{-1}(N_{s_0})$. Thus, K_{q_0} is a I_{q_0}-invariant complex curve in M' equivalent to either an annulus or a punctured disk. By Bochner's linearization theorem (see [Bo]) there exist a local holomorphic change of coordinates

F near q_0 on M that identifies an I_{q_0}-invariant neighborhood U of q_0 with an L_{q_0}-invariant neighborhood of the origin in $T_{q_0}(M)$ such that $F(q_0) = 0$ and $F(gq) = \alpha_{q_0}(g)F(q)$ for all $g \in I_{q_0}$ and $q \in U$. It follows from (iii) of Proposition 3.2 that L_{q_0} has two invariant subspaces in $T_{q_0}(M)$. One of them corresponds in our coordinates to O, the other to a complex curve C intersecting O transversally at q_0. Since near q_0 the curve C coincides with $K_{q_0} \cup \{q_0\}$, in a neighborhood of q_0 the curve K_{q_0} is equivalent to a punctured disk. Further, if a sequence $\{q_j\}$ from K_{q_0} accumulates to q_0, the sequence $\{f(q_j)\}$ accumulates to one of the two ends of N_{s_0}, and therefore we have either $r = 0$ or $R = \infty$. Since both these conditions cannot be satisfied simultaneously due to the hyperbolicity of M, we conclude that O is the only complex hypersurface orbit in M.

Assume first that $r = 0$. We extend f to a map from M onto the domain

$$\left\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'|^2 + \frac{1}{R^\alpha}|z_n|^\alpha < 1\right\} \tag{3.30}$$

by setting $f(q_0) = s_0$, where $q_0 \in O$ and $s_0 \in c_0$ are related as specified above. The extended map is one-to-one and satisfies (3.16) for all $g \in G(M)$, $q \in M$. To prove that f is holomorphic on all of M, it suffices to show that f is continuous on O. We will prove that every sequence $\{q_j\}$ in M converging to q_0 has a subsequence along which the values of f converge to s_0. Let first $\{q_j\}$ be a sequence in O. Clearly, there exists a sequence $\{g_j\}$ in $G(M)$ such that $q_j = g_j q_0$ for all j. Since $G(M)$ acts properly on M, there exists a converging subsequence $\{g_{j_k}\}$ of $\{g_j\}$, and we denote by g_0 its limit. It then follows that $g_0 \in I_{q_0}$ and, since f satisfies (3.16), we obtain that $\{f(q_{j_k})\}$ converges to s_0. Next, if $\{q_j\}$ is a sequence in M', then there exists a sequence $\{g_j\}$ in $G(M)$ such that $g_j q_j \in K_{q_0}$. Clearly, the sequence $\{g_j q_j\}$ converges to q_0 and hence $\{f(g_j q_j)\}$ converges to s_0. Again, the properness of the $G(M)$-action on M yields that there exists a converging subsequence $\{g_{j_k}\}$ of $\{g_j\}$. Let g_0 be its limit; as before, we have $g_0 \in I_{q_0}$. Property (3.16) now implies $f(q_{j_k}) = [\varphi(g_{j_k})]^{-1} f(g_{j_k} q_{j_k})$, and therefore $\{f(q_{j_k})\}$ converges to s_0. Thus, f is holomorphic on M, and therefore M is holomorphically equivalent to domain (3.30) and hence to the domain $\mathfrak{E}_\alpha$ (see (3.2)). Clearly, $d(\mathfrak{E}_\alpha) = n^2$ only if $\alpha \neq 2$.

Assume now that $R = \infty$. Observe that one can extend the action of the group R_{ε_α} on $\mathcal{C}$ to an action on $\tilde{\mathcal{C}} := \mathbb{B}^{n-1} \times \mathbb{CP}^1$ by holomorphic transformations by setting $g(z', \infty) := (a(z'), \infty)$ for every $g \in R_{\varepsilon_\alpha}$, where a is the corresponding automorphism of $\mathbb{B}^{n-1}$ in the z'-variables (see formula (3.1)). Now arguing as in the case $r = 0$, we can extend f to a biholomorphic map between M and the domain in $\tilde{\mathcal{C}}$

$$\begin{aligned}\Big\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'| < 1,\, |z_n| > r(1 - |z'|^2)^{1/\alpha}\Big\}& \\ \cup \left(\mathbb{B}^{n-1} \times \{\infty\}\right).&\end{aligned} \tag{3.31}$$

This domain is holomorphically equivalent to $\mathcal{E}_{-1/\alpha}$ (see (3.2)), and so is M.

In the case when M' is holomorphically equivalent to C_r^R for some $0 \le r < R \le \infty$, a similar argument gives that M has to be equivalent to $\mathbb{B}^{n-1} \times \Delta$, which is impossible.

It now remains to show that all manifolds in (i)-(vii) of (3.2) are pairwise holomorphically non-equivalent. Since the automorphism groups of most manifolds are non-isomorphic (see the discussion following the formulation of Theorem 3.1 in Sect. 3.1), and the orbits of $U_n/\mathbb{Z}_m$ in $\mathbb{C}^n \setminus \{0\}/\mathbb{Z}_m$ are topologically different for distinct m, we must only prove pairwise non-equivalence of domains within each of the following families: $\{$(ii)-(iv)$\}$, $\{$(v)$\}$, $\{$(vi)$\}$. The first two families consist of Reinhardt domains. It is shown in [Kru] that two hyperbolic Reinhardt domains are holomorphically equivalent if and only if they are equivalent by means of an elementary algebraic map, that is, a map of the form

$$z_i \mapsto \lambda_i z_1^{a_{i1}} \cdot \ldots \cdot z_n^{a_{in}}, \quad i = 1, \ldots, n,$$

where $\lambda_i \in \mathbb{C}^*$, $a_{ij} \in \mathbb{Z}$ for all i, j, and $\det(a_{ij}) = \pm 1$. It is straightforward to verify, however, that no two domains within the first two families are algebraically equivalent.

Next, for every domain in the third family its group of holomorphic automorphisms is the group R_ω. Therefore, a biholomorphic map between any two domains D_1 and D_2 in this family takes every R_ω-orbit from D_1 into an R_ω-orbit in D_2. However, as we noted above, every CR-isomorphism between two R_ω-orbits is a composition of an element of R_ω and a translation in z_n. Therefore, D_1 can be mapped onto D_2 by such a translation. It is clear, however, that no two domains in the third family can be obtained from one another by translating z_n.

This completes the proof of Theorem 3.1. ■

4

The Case $d(M) = n^2 - 1$, $n \geq 3$

In this chapter we begin classifying connected non-homogeneous hyperbolic manifolds of dimension $n \geq 2$ with $d(M) = n^2 - 1$. We start with the case $n \geq 3$. The far more involved case $n = 2$ will be considered in Chap. 5. We will now state the main result of the present chapter.

4.1 Main Result

Theorem 4.1. ([I3]) *Let M be a connected non-homogeneous hyperbolic manifold of dimension $n \geq 3$ with $d(M) = n^2 - 1$. Then M is holomorphically equivalent to $\mathbb{B}^{n-1} \times S$, where S is a hyperbolic Riemann surface with $d(S) = 0$.*

The proof of Theorem 4.1 goes along the lines of the proof of Theorem 3.1 in Chap. 3. First of all, in Sect. 4.2 we obtain an initial classification of $G(M)$-orbits in M. It turns out that every $G(M)$-orbit is either a real or complex hypersurface in M, every real hypersurface orbit is spherical and every complex hypersurface orbit is holomorphically equivalent to $\mathbb{B}^{n-1}$ (see Proposition 4.2). Note that Proposition 4.2 also contains important information about $G(M)$-orbits for $n = 2$, in particular, it allows in this case for real hypersurface orbits to be either Levi-flat or Levi non-degenerate non-spherical, and for 2-dimensional orbits to be totally real rather than complex submanifolds of M. It turns out that all these possibilities indeed realize (see Chap. 5).

Next, in Sect. 4.3 we show that real hypersurface orbits in fact cannot occur (see Theorem 4.3). First, we prove that there may be five possible kinds of such orbits (identical to those found in Proposition 3.3 in Chap. 3) and that the presence of an orbit of a particular kind determines $G(M)$ as a Lie group. Further, when we attempt to glue real hypersurface orbits together as we did in the proof of Theorem 3.1 in Sect. 3.4, it turns out that for any resulting hyperbolic manifold M, the value of $d(M)$ is always greater than

$n^2 - 1$. Hence all $G(M)$-orbits are in fact complex hypersurfaces. The proof of Theorem 4.1 is finalized in Sect. 4.4.

4.2 Initial Classification of Orbits

In this section we prove the following proposition.

Proposition 4.2. ([I3]) *Let M be a connected non-homogeneous hyperbolic manifold of dimension $n \geq 2$ with $d(M) = n^2 - 1$. Fix $p \in M$ and let $V := T_p(O(p))$. Then*

(i) the orbit $O(p)$ is either a real or complex closed hypersurface in M, or, for $n = 2$, a totally real 2-dimensional closed submanifold of M;

(ii) if $O(p)$ is a real hypersurface, it is either strongly pseudoconvex or, for $n = 2$, Levi-flat and foliated by complex curves holomorphically equivalent to the unit disk Δ; if $O(p)$ is strongly pseudoconvex and $n \geq 3$, then $O(p)$ is spherical; there exist coordinates in $T_p(M)$ such that – with respect to the orthogonal decomposition $T_p(M) = (V \cap iV)^{\perp} \oplus (V \cap iV)$ – the group L_p is either a subgroup of $\{id\} \times U_{n-1}$, or, if $n = 2$ and $O(p)$ is Levi-flat, a finite subgroup of $\mathbb{Z}_2 \times U_1$; moreover, $L_p^0 = \{id\} \times SU_{n-1}$;

(iii) if $O(p)$ is a complex hypersurface, it is holomorphically equivalent to $\mathbb{B}^{n-1}$; in this case, if $n \geq 3$, there exist coordinates $(z_1, \ldots, z_n)$ in $T_p(M)$ in which $V = \{z_1 = 0\}$ and L_p^0 is given by the group $H^n_{k_1,k_2}$ of all matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix}, \tag{4.1}$$

where $B \in U_{n-1}$ and $a \in (\det B)^{\frac{k_1}{k_2}}$, for some $k_1, k_2 \in \mathbb{Z}$, $(k_1, k_2) = 1$, $k_2 \neq 0$; furthermore, if $n = 2$, there exist holomorphic coordinates (z, w) in $T_p(M)$ in which $V = \{z = 0\}$ and L_p^0 is given by either $H^2_{k_1,k_2}$ for some k_1, k_2, or the group of matrices

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.2}$$

where $|a| = 1$;

(iv) if $n = 2$ and $O(p)$ is totally real, then $T_p(M) = V \oplus iV$, and there exist coordinates in V such that every transformation from L_p^0 has the form: $v_1 + iv_2 \mapsto Av_1 + iAv_2$, $v_1, v_2 \in V$, where $A \in SO_2(\mathbb{R})$.

Proof: Again, we consider the three cases as in the proof of Proposition 2.1 (cf. the proof of Proposition 3.2).

Case 1. $d := \dim_{\mathbb{C}}(V + iV) < n$.

Arguing as in the proof of Proposition 2.1 we obtain

$$n^2 - 1 \leq (n-d)^2 + d^2 + 2d,$$

which yields that either $d = 0$ or $d = n - 1$. If $d = 0$, then $I_p = G(M)$ and hence L_p is isomorphic to $G(M)$. Therefore, $\dim L_p = n^2 - 1$, which implies that $L_p = SU_n$. The group SU_n acts transitively on real directions in $T_p(M)$, and it follows from Theorem 1.7 that M is holomorphically equivalent to $\mathbb{B}^n$. This is clearly impossible, hence $d = n - 1$. Then we have

$$n^2 - 1 = \dim L_p + \dim O(p) \leq n^2 - 2n + 2 + \dim O(p).$$

Thus, $\dim O(p) \geq 2n - 3$, that is, either $\dim O(p) = 2n - 2$, or $\dim O(p) = 2n - 3$.

Suppose first that $\dim O(p) = 2n - 2$. In this case we have $iV = V$, and therefore $O(p)$ is a complex hypersurface in M. Since $\dim L_p = (n-1)^2$, it follows from the proof of Lemma 2.1 of [IKru1] that L_p^0 is either $U_1 \times SU_{n-1}$, or, for some k_1, k_2, the group $H^n_{k_1,k_2}$ defined in (4.1). Therefore, if $n \geq 3$ or $n = 2$ and $L_p^0 = H^2_{k_1,k_2}$ for some k_1, k_2, then L_p acts transitively on real directions in V, and Theorem 1.7 implies that $O(p)$ is holomorphically equivalent to $\mathbb{B}^{n-1}$.

Let $n \geq 3$ and $L_p^0 = U_1 \times SU_{n-1}$. It then follows from [Bo] (see also [Ka]) that $I'_p := \alpha_p^{-1}(U_1)$ is the kernel of the action of $G(M)$ on $O(p)$; in particular, I'_p is normal in $G(M)$. Therefore, the quotient $G(M)/I'_p$ acts effectively on $O(p)$. Clearly, $\dim G(M)/I'_p = n^2 - 2$. Thus, the group $\mathrm{Aut}(O(p))$ is isomorphic to $\mathrm{Aut}(\mathbb{B}^{n-1})$ (in particular, its dimension is $n^2 - 1$) and has a codimension 1 subgroup. However, the Lie algebra $\mathfrak{su}_{n-1,1}$ of the group $\mathrm{Aut}(\mathbb{B}^{n-1})$ does not have codimension 1 subalgebras, if $n \geq 3$ (see, e.g., [EaI]). Thus, we have shown that if $n \geq 3$, then $L_p^0 = H^n_{k_1,k_2}$ for some k_1, k_2.

Next, if $n = 2$ and $L_p^0 = U_1 \times \{\mathrm{id}\}$ (see (4.2)), then the above argument shows that $O(p)$ is a hyperbolic complex curve with automorphism group of dimension at least 2. Hence $O(p)$ is holomorphically equivalent to Δ.

Suppose now that $\dim O(p) = 2n - 3$. In this case $\dim L_p = n^2 - 2n + 2$. which implies $L_p = U_1 \times U_{n-1}$. In particular, L_p acts transitively on real directions in $V + iV$. This is, however, impossible since V is of codimension 1 in $V + iV$ and is L_p-invariant.

Case 2. $T_p(M) = V + iV$ and $r := \dim_{\mathbb{C}}(V \cap iV) > 0$.

In this case we obtain

$$n^2 - 1 < (n-r)^2 + r^2 + 2n - 1,$$

which shows that either $r = 1$, or $r = n - 1$. It then follows that $\dim L_p < n^2 - 2n + 2$. Therefore, we have

$$n^2 - 1 = \dim L_p + \dim O(p) < n^2 - 2n + 2 + \dim O(p).$$

Hence $\dim O(p) > 2n - 3$. Thus, we have either $\dim O(p) = 2n - 1$, or $\dim O(p) = 2n - 2$.

Suppose that $\dim O(p) = 2n - 1$. As in the proof of Proposition 3.2, we consider the orthogonal complement W to $T_p^c(O(p)) = V \cap iV$ in $T_p(M)$. Since $r = n - 1$, we have $\dim_{\mathbb{C}} W = 1$. The group L_p is a subgroup of U_n and preserves V, $T_p^c(O(p))$, and W; hence it preserves the line $W \cap V$. Therefore, it can act only as $\pm\text{id}$ on W, that is, $L_p \subset \mathbb{Z}_2 \times U_{n-1}$. Since $\dim L_p = (n-1)^2 - 1$, we have $L_p^0 = \{\text{id}\} \times SU_{n-1}$. In particular, L_p acts transitively on real directions in $T_p^c(O(p))$, if $n \geq 3$. Hence, the orbit $O(p)$ is either Levi-flat or strongly pseudoconvex for all $n \geq 2$. If $O(p)$ is strongly pseudoconvex, then $L_p \subset \{\text{id}\} \times U_{n-1}$.

Suppose first that $n \geq 3$ and $O(p)$ is Levi-flat. Then $O(p)$ is foliated by connected complex submanifolds. Consider, as before, the leaf $O(p)_p$ passing through p. As in the proof of Proposition 3.5, denote by $\mathfrak{g}(M)$ the Lie algebra of $G(M)$-vector fields on M and identify it with the Lie algebra of $G(M)$. We further identify every vector field from $\mathfrak{g}(M)$ with its restriction to $O(p)$. Let $\mathfrak{l}_p \subset \mathfrak{g}(M)$ be the subspace of all vector fields tangent to $O(p)_p$ at p. Since vector fields in $\mathfrak{l}_p$ remain tangent to $O(p)_p$ at each point in $O(p)_p$, the subspace $\mathfrak{l}_p$ is in fact a Lie subalgebra of $\mathfrak{g}(M)$. It follows from the definition of $\mathfrak{l}_p$ that $\dim \mathfrak{l}_p = n^2 - 2$. As in the proof of Proposition 3.5, we denote by H_p the (possibly non-closed) connected subgroup of $G(M)$ with Lie algebra $\mathfrak{l}_p$. It is straightforward to verify that the group H_p acts on $O(p)_p$ by holomorphic transformations and that $I_p^0 \subset H_p$. If some non-trivial element $g \in H_p$ acts trivially on $O(p)_p$, then $g \in I_p$, and it corresponds to the non-trivial element in $\mathbb{Z}_2$ (recall that $L_p \subset \mathbb{Z}_2 \times U_{n-1}$). As in the proof of Proposition 3.5, we denote such g by g_p. Thus, either H_p or $H_p/\mathbb{Z}_2$ acts effectively on $O(p)_p$ (the former case occurs if $g_p \notin H_p$, the latter if $g_p \in H_p$). The group L_p acts transitively on real directions in the tangent space $T_p^c(O(p))$ to $O(p)_p$, and it follows from Theorem 1.7 that $O(p)_p$ is holomorphically equivalent to $\mathbb{B}^{n-1}$. Therefore, the group $\text{Aut}(O(p)_p)$ is isomorphic to $\text{Aut}(\mathbb{B}^{n-1})$ (in particular, its dimension is $n^2 - 1$) and has a codimension 1 (possibly non-closed) subgroup. However, as we noted above, the Lie algebra of $\text{Aut}(\mathbb{B}^{n-1})$ does not have codimension 1 subalgebras, if $n \geq 3$. Thus, $O(p)$ is in fact strongly pseudoconvex. Since $L_p^0 = \{\text{id}\} \times SU_{n-1}$, we have $d_p(O(p)) \geq (n-1)^2 - 1$, and Propositions 1.5, 1.6 give that $O(p)$ is spherical for $n \geq 3$.

For $n = 2$ the above argument shows that $O(p)$ is foliated by connected hyperbolic complex curves with automorphism group of dimension at least 2, that is, by complex curves holomorphically equivalent to Δ.

Suppose now that $\dim O(p) = 2n - 2$. Since $T_p(M) = V + iV$, we have $V \neq iV$. Therefore, we have $r = n - 2$, which is only possible for $n = 3$ (recall that we have either $r = 1$, or $r = n - 1$). In this case $\dim L_p = 4$ and, arguing as in the proof of Lemma 2.1 of [IKru1], we see that L_p acts transitively on real directions in the orthogonal complement W to $T_p^c(O(p))$ in $T_p(M)$. This is, however, impossible since L_p must preserve $W \cap V$.

Case 3. $T_p(M) = V \oplus iV$.

In this case for $n \geq 3$ we obtain a contradiction as in the proof of Proposition 2.1. If $n = 2$, we clearly have $\dim L_p = 1$. Choosing coordinates in V in which $L_p^0|_V = SO_2(\mathbb{R})$, we obtain that L_p^0 acts on $T_p(M)$ as $v_1 + iv_2 \mapsto Av_1 + iAv_2$, $v_1, v_2 \in V$, where $A \in SO_2(\mathbb{R})$.

The proof of the proposition is complete. ■

4.3 Non-Existence of Real Hypersurface Orbits

In this section we deal with real hypersurface orbits and eventually show that they do not in fact occur. Our goal is to prove the following theorem.

Theorem 4.3. ([I3]) *Let M be a connected non-homogeneous hyperbolic manifold of dimension $n \geq 3$ with $d(M) = n^2 - 1$. Then no orbit in M is a real hypersurface.*

Proof: First we narrow down the class of all possible real hypersurface orbits by proving the following analogue of Proposition 3.3.

Proposition 4.4. ([I3]) *Let M be a connected hyperbolic manifold of dimension $n \geq 3$ with $d(M) = n^2 - 1$. Assume that for a point $p \in M$ its orbit $O(p)$ is a real hypersurface. Then $O(p)$ is CR-equivalent to one of the hypersurfaces listed in (3.3).*

Proof of Proposition 4.4: The proof is similar to that of Proposition 3.3. Recall that every real hypersurface orbit in M is spherical (see (ii) of Proposition 4.2). Excluding from the list of all homogeneous domains in S^{2n-1} given in Theorem 3.1 of [BS1] those with automorphism group of dimension less than $n^2 - 1$ again leads to domains (3.5). Hence, as before, $\tilde{O}(p)$ is one of manifolds (3.6).

If $\tilde{O}(p) = S^{2n-1}$, then, utilizing as before Proposition 5.1 of [BS1], we obtain that the orbit $O(p)$ is CR-equivalent to a lens space as in (i) of (3.3).

Suppose next that $\tilde{O}(p) = \sigma$. The group $\mathrm{Aut}_{CR}(\sigma)$ consists of all maps of the form (3.7) and has dimension $n^2 + 1$. In particular, we have $n^2 - 1 \leq \dim G \leq n^2 + 1$. If $\dim G = n^2 + 1$, we obtain, as in the proof of Proposition 3.3, that $O(p)$ is CR-equivalent to σ.

Assume now that $n^2 - 1 \leq \dim G \leq n^2$. Since G acts transitively on σ, we have $N \subset G$, where N is the Heisenberg group (cf. the proof of Proposition

3.3). As before, this yields that $O(p)$ is CR-equivalent to the hypersurface δ (see (3.3)).

Suppose finally that $\tilde{O}(p) = \omega$. The group $\mathrm{Aut}_{CR}(\omega)$ consists of all maps of the form (3.11) and has dimension n^2. In particular, $n^2 - 1 \leq \dim G \leq n^2$. Thus, either $G = \mathrm{Aut}_{CR}(\omega)$, or G coincides with the subgroup of $\mathrm{Aut}_{CR}(\omega)$ given by the condition $\beta = 0$ in formula (3.11). In either case, the centralizer of G in $\mathrm{Aut}_{CR}(\omega)$ consists of all maps of the form (3.8). As before, this gives that $O(p)$ is CR-equivalent to either ω or ε_α for some $\alpha > 0$.

The proof of Proposition 4.4 is complete. ■

Suppose that for some $p \in M$ the orbit $O(p)$ is a real hypersurface in M, and let $\mathfrak{m}$ be the manifold from list (3.3) to which $O(p)$ is CR-equivalent (as in Chap. 3, we say that $\mathfrak{m}$ is the model for $O(p)$). Since $G(M)$ acts properly and effectively on $O(p)$, the CR-equivalence induces an isomorphism between $G(M)$ and a closed connected (n^2-1)-dimensional subgroup $R_{\mathfrak{m}}$ of $\mathrm{Aut}_{CR}(\mathfrak{m})$, that acts properly and transitively on $\mathfrak{m}$. We will now prove an analogue of Proposition 3.4.

Proposition 4.5. ([I3])

(i) $R_{S^{2n-1}}$ is conjugate to SU_n in $\mathrm{Aut}_{CR}(S^{2n-1})$, and $R_{\mathcal{L}_m} = SU_n/(SU_n \cap \mathbb{Z}_m)$ for $m > 1$;
(ii) R_σ consists of all maps of the form (3.7) with $\lambda = 1$ and $U \in SU_{n-1}$;
(iii) R_δ consists of all maps of the form (3.13) with $U \in SU_{n-1}$;
(iv) R_ω consists of all maps of the form (3.11) with $\beta = 0$;
(v) ε_α can only be a model for an orbit in M if $\alpha \in \mathbb{Q}$, and in this case the group R_{ε_α} consists of all maps of the form (3.1) with $\theta = 1/\alpha$ and $\beta = 0$.

Proof of Proposition 4.5: The proof is similar to that of Proposition 3.4. If $\mathfrak{m} = \mathcal{L}_m$ for some $m \in \mathbb{N}$, then $G(M)$ is compact. Assume first that $m = 1$. In this case $R_{S^{2n-1}}$ is a subgroup of $\mathrm{Aut}_{CR}(S^{2n-1}) = \mathrm{Aut}(\mathbb{B}^n)$. Since $R_{S^{2n-1}}$ is compact, it is conjugate to a subgroup of U_n, which is a maximal compact subgroup in $\mathrm{Aut}(\mathbb{B}^n)$. Since $R_{S^{2n-1}}$ is (n^2-1)-dimensional, it is conjugate to SU_n. Suppose now that $m > 1$. Recall that the group $\mathrm{Aut}_{CR}(\mathcal{L}_m)$ coincides with $U_n/\mathbb{Z}_m$ acting on $\mathbb{C}^n \setminus \{0\}/\mathbb{Z}_m$ in the standard way. Since $R_{\mathcal{L}_m}$ is of codimension 1 in $\mathrm{Aut}_{CR}(\mathcal{L}_m)$, we have $R_{\mathcal{L}_m} = SU_n/(SU_n \cap \mathbb{Z}_m)$, and (i) of the proposition is obtained.

Assume now that $\mathfrak{m} = \sigma$. Since I_p^0 is isomorphic to SU_{n-1}, the connected identity component of the isotropy subgroup of the origin under the R_σ-action on σ coincides with the group of all maps of the form (3.7) with $U \in SU_{n-1}$, $\lambda = 1$, $a = 0$, $\alpha = 0$. Furthermore, since R_σ acts transitively on σ, it contains the Heisenberg group N, and (ii) of the proposition follows.

Next, the group $\mathrm{Aut}_{CR}(\delta)$ consists of all maps of the form (3.13) and has dimension n^2. In this case the connected identity component of the isotropy subgroup of the point $(0,\dots,0,1)$ under the R_δ-action on δ coincides with the group of all maps of the form (3.13) with $U \in SU_{n-1}$, $a=0$, $\beta=0$. The transitivity of the action now gives (iii) of the proposition.

Further, the group $\mathrm{Aut}_{CR}(\omega)$ consists of all maps of the form (3.11) and has dimension n^2. Its only codimension 1 subgroup is given by the condition $\beta=0$, which yields (iv) of the proposition.

Finally, the group $\mathrm{Aut}_{CR}(\varepsilon_\alpha)$ consists of all maps of the form (3.1) with $\theta=1/\alpha$ and has dimension n^2. It has a unique codimension 1 subgroup; this subgroup is given by the condition $\beta=0$. Observe now that it is closed in $\mathrm{Aut}_{CR}(\varepsilon_\alpha)$ only if $\alpha \in \mathbb{Q}$, and (v) of the proposition follows.

The proof of Proposition 4.5 is complete. ■

We will now finish the proof of Theorem 4.3. Our argument is similar to that given in Sect. 3.4.

If for some $p \in M$ the model for $O(p)$ is S^{2n-1}, it follows from Proposition 4.5 that M admits an effective action of SU_n by holomorphic transformations and therefore is holomorphically equivalent to one of the manifolds listed in [IKru2] (see Sect. 6.3). However, none of the manifolds on this list with $n \geq 3$ is hyperbolic and has an (n^2-1)-dimensional automorphism group. Thus, S^{2n-1} is not the model for any real hypersurface orbit in M.

As before, we now observe that for every model $\mathfrak{m}$ the group $R_{\mathfrak{m}}$ acts by holomorphic transformations with real hypersurface orbits on a manifold $M_{\mathfrak{m}}$ containing $\mathfrak{m}$, such that every orbit of the $R_{\mathfrak{m}}$-action on $M_{\mathfrak{m}}$ is equivalent to $\mathfrak{m}$ by means of a holomorphic automorphism of $M_{\mathfrak{m}}$ of a simple form. In addition, every element of $\mathrm{Aut}_{CR}(\mathfrak{m})$ extends to a holomorphic automorphism of $M_{\mathfrak{m}}$. The manifolds $M_{\mathfrak{m}}$ and corresponding automorphisms coincide with those listed under (a)–(e) in Sect. 3.4.

We now assume that every orbit in M is a real hypersurface and apply the orbit gluing procedure introduced in Sect. 3.4. Compared to the situation considered there, the only difference arises at step (II). We will show that the extended map $\mathfrak{F}$ is 1-to-1 on an $R_{\mathfrak{m}}$-invariant neighborhood of $\mathfrak{m}$ contained in U'. Suppose that for some $s_0, s_0' \in U'$, $s_0 \neq s_0'$, we have $\mathfrak{F}(s_0)=\mathfrak{F}(s_0')$. This can only occur if s_0 and s_0' lie in the same $R_{\mathfrak{m}}$-orbit, and therefore there exist a point $s \in U$ and elements $h, h' \in R_{\mathfrak{m}}$ such that $s_0 = hs$, $s_0' = h's$. Then we have $h'^{-1}h \notin J_s$ and $\varphi^{-1}(h'^{-1}h) \in I_{\mathfrak{F}(s)}$, where J_s denotes the isotropy subgroup of s under the $R_{\mathfrak{m}}$-action. At the same time, we have as before $\varphi^{-1}(J_s) \subset I_{\mathfrak{F}(s)}$. It follows from the explicit forms of the models and the corresponding groups that the connected identity components of $I_{\mathfrak{F}(s)}$ and J_s are isomorphic to SU_{n-1}, and that $I_{\mathfrak{F}(s)}$ has more connected components than J_s only if $\mathfrak{m} = \varepsilon_{m/k_1}$, $\mathcal{O}(s)$ is equivalent to ε_{m/k_1}, and the model for $O(\mathfrak{F}(s))$ is ε_{m/k_2} for some $m, k_1, k_2 \in \mathbb{N}$, $(m,k_1)=1$, $(m,k_2)=1$, $k_2 > k_1$. If there is a neighborhood of p not containing a point q such that the model for $O(q)$ is some $\varepsilon_{m/k}$, with $k \in \mathbb{N}$, $(m,k)=1$, $k > k_1$, then $\mathfrak{F}$ is biholomorphic on an $R_{\mathfrak{m}}$-invariant open subset of U' containing $\mathfrak{m}$. Suppose now that in every neighborhood of p (that we assume to be contained in W) there is a point

q such that the model for $O(q)$ is $\varepsilon_{m/k}$, with $k \in \mathbb{N}$, $(m, k) = 1$, $k > k_1$. Choose a sequence of such points $\{q_j\} \subset W$ converging to p. Since the action of $G(M)$ on M is proper, the isotropy subgroups I_{q_j} converge to I_p. Every subgroup I_{q_j} has more connected components than $J_{f(q_j)}$, and therefore for a subsequence $\{j_k\}$ of the sequence of indices $\{j\}$ there is a sequence $\{g_{j_k}\}$ with $g_{j_k} \in I_{q_{j_k}}$, $\varphi(g_{j_k}) \notin J_{f(q_{j_k})}$, such that $\{g_{j_k}\}$ converges to an element of I_p. At the same time, the sequence $\{f(q_j)\}$ converges to $f(p)$ and therefore $\varphi(g_{j_k})f(q_{j_k})$ lies in U for large k. For large k we have $\mathfrak{F}(\varphi(g_{j_k})f(q_{j_k})) = q_{j_k}$. Since $\varphi(g_{j_k}) \notin J_{f(q_{j_k})}$, the point $\varphi(g_{j_k})f(q_{j_k})$ does not coincide with $f(q_{j_k})$. Thus, we have found two distinct points in U (namely, $f(q_{j_k})$ and $\varphi(g_{j_k})f(q_{j_k})$ for large k) mapped by $\mathfrak{F}$ into the same point in W, which contradicts the fact that $\mathfrak{F}$ is 1-to-1 on U. Hence, as in Sect. 3.4, we have shown that f can be extended to a biholomorphic map satisfying (3.16) between a $G(M)$-invariant neighborhood of $O(p)$ in M and a $R_{\mathfrak{m}}$-invariant neighborhood of $\mathfrak{m}$ in $M_{\mathfrak{m}}$.

Applying the orbit gluing procedure, we obtain, as in Sect. 3.4, that M is holomorphically equivalent to some $S_r/\mathbb{Z}_m$ (see (2.1)) in case (a), to either $\mathfrak{C}$ or $\mathbb{B}^n$ in case (b), to either some $\mathfrak{A}_{r,1}$ or $\mathfrak{A}_{0,-1}$ in case (c), to either some $\mathfrak{B}_{r,1}$ or $\mathfrak{B}_{0,-1}$ in case (d), and to either some $\mathcal{E}_{r,1/\alpha}$ or $\mathcal{E}_{0,-1/\alpha}$ in case (e) – see (3.2) for the definitions of these domains. However, the dimension of the automorphism group of each of these manifolds is at least n^2.

Assume now that both real and complex hypersurface orbits are present in M. Since the action of $G(M)$ on M is proper, it follows as in the proof of Proposition 3.2, that there are at most two complex hypersurface orbits in M. Statement (iii) of Proposition 4.2 gives that if a complex hypersurface orbit is present in M, then the quotient of $G(M)$ by a finite normal subgroup is isomorphic to $\mathrm{Aut}(\mathbb{B}^{n-1})$. Proposition 4.5 now yields that the model for every real hypersurface orbit in M is ε_α for some $\alpha > 0$, $\alpha \in \mathbb{Q}$.

Let M', as before, be the manifold obtained from M by removing all complex hypersurface orbits. It then follows from the above considerations that M' is holomorphically equivalent to $\mathcal{E}^R_{r,\alpha}$ for some $0 \leq r < R \leq \infty$ (see (3.23)) by means of a map f satisfying (3.16) for all $g \in G(M)$, $q \in M'$ and some isomorphism $\varphi : G(M) \to R_{\varepsilon_\alpha}$. The group R_{ε_α} acts on all of $\mathcal{C}$ (see Sect. 3.4), and the orbit of any point in $\mathcal{C}$ with $z_n = 0$ is the complex hypersurface c_0 (see (3.26)). For a point $s \in \mathcal{C}$ denote by J_s the isotropy subgroup of s under the action of R_{ε_α}. If $s_0 \in c_0$ and $s_0 = (z_0', 0)$, then J_{s_0} is isomorphic to $H^n_{k_1,k_2}$, where $k_1/k_2 = 2/\alpha n$ and consists of all elements of R_{ε_α} for which the transformations in the z'-variables form the isotropy subgroup of the point z_0' in $\mathrm{Aut}(\mathbb{B}^{n-1})$.

Fix $s_0 = (z_0', 0) \in c_0$ and consider the set N_{s_0} defined as in (3.27). Clearly, N_{s_0} is given by formula (3.28), and therefore is either an annulus (possibly, with infinite outer radius) or a punctured disk. In particular, N_{s_0} is a complex curve in $\mathcal{C}_0$ (see Sect. 3.4).

Since J_{s_0} is a maximal compact subgroup of R_{ε_α}, the group $\varphi^{-1}(J_{s_0})$ is a maximal compact subgroup of $G(M)$. Let O be a complex hypersurface orbit in M. For $q \in O$ the isotropy subgroup I_q is compact and has dimension $(n-1)^2 = \dim J_{s_0}$. It then follows that I_q is a maximal compact subgroup of $G(M)$ as well (in particular, I_q is connected), and hence $\varphi^{-1}(J_{s_0})$ is conjugate to I_q for every $q \in O$. Therefore, there exists $q_0 \in O$ such that $\varphi^{-1}(J_{s_0}) = I_{q_0}$. Since the isotropy subgroups in R_{ε_α} of distinct points in c_0 do not coincide, such a point q_0 is unique.

Now, arguing as in Sect. 3.4, one can show that f extends to a biholomorphic map from M onto one of domains (3.30), (3.31). This is, however, impossible, since the automorphism group of each of these domains has dimension at least n^2.

The proof of Theorem 4.3 is complete. ■

4.4 Proof of Theorem 4.1

By Theorem 4.3, the orbit of every point in M is a complex hypersurface. Fix $p \in M$. By (iii) of Proposition 4.2 the group L_p^0 is given by some $H^n_{k_1,k_2}$ (see (4.1)). Observe that if $k_1 \neq 0$, then there exists a neighborhood U of p such that for every $q \in U \setminus O(p)$ the values at q of $G(M)$-vector fields span a codimension 1 subspace of $T_q(M)$. Hence a real hypersurface orbit is present in M in this case. This contradiction shows that in fact $k_1 = 0$, that is, $L_p^0 = H^n_{0,1}$. Arguing as in the proof of Lemma 4.4 of [IKru1], we now obtain that the full group L_p consists of all matrices of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & B \end{pmatrix}, \tag{4.3}$$

where $B \in U_{n-1}$ and $\alpha^m = 1$ for some $m \geq 1$. It then follows from [Bo] that the kernel of the action of $G(M)$ on $O(p)$ is $I'_p := \alpha_p^{-1}(\mathbb{Z}_m)$, where we identify $\mathbb{Z}_m$ with the subgroup of L_p that consists of all matrices of the form (4.3) with $B = \mathrm{id}$. Thus, $G(M)/I'_p$ acts effectively on $O(p)$. Since $O(p)$ is holomorphically equivalent to $\mathbb{B}^{n-1}$ and $\dim G(M) = n^2 - 1 = d(\mathbb{B}^{n-1})$, we obtain that $G(M)/I'_p$ is isomorphic to $\mathrm{Aut}(\mathbb{B}^{n-1})$. Since the image of I_p under the projection $G(M) \to \mathrm{Aut}(\mathbb{B}^{n-1})$ is a maximal compact subgroup of $\mathrm{Aut}(\mathbb{B}^{n-1})$, it follows that I_p is a maximal compact subgroup in $G(M)$. However, every maximal compact subgroup of a connected Lie group is connected whereas I_p is not if $m > 1$. Thus, $m = 1$, and hence $G(M)$ is isomorphic to $\mathrm{Aut}(\mathbb{B}^{n-1})$. It also follows that L_p fixes every point in the orthogonal complement W to $T_p(O(p))$ in $T_p(M)$. Observe that the above arguments apply to every point in M.

Define

$$S_p := \{s \in M : I_s = I_p\}.$$

Clearly, I_p fixes every point in S_p and $N_{gp} = gS_p$ for all $g \in G(M)$. Further, since for two distinct points s_1, s_2 lying in the same orbit we have $I_{s_1} \neq I_{s_2}$, the set S_p intersects every orbit in M at exactly one point. By Bochner's linearization theorem (see [Bo]) there exist a local holomorphic change of coordinates F near p in M that identifies an I_p-invariant neighborhood U of p with an L_p-invariant neighborhood $\mathcal{U}$ of the origin in $T_p(M)$ such that $F(p) = 0$ and $F(gq) = \alpha_p(g)F(q)$ for all $g \in I_p$ and $q \in U$. Since $L_p = H^n_{0,1}$, we have $S_p \cap U = F^{-1}(W \cap \mathcal{U})$. In particular, S_p is a complex curve near p. Since the same argument can be carried out at every point of S_p, we obtain that S_p is a closed complex hyperbolic curve in M.

We will now construct a biholomorphic map $f : M \to \mathbb{B}^{n-1} \times S_p$. Let $f_0 : O(p) \to \mathbb{B}^{n-1}$ be a biholomorphism. For $q \in M$ let r be the (unique) point where S_p intersects $O(q)$. Let $g \in G(M)$ be such that $q = gr$. Then we set $f(q) := (f_0(gp), r)$. By construction, f is biholomorphic. Clearly, we have $d(S_p) = 0$.

The proof of Theorem 4.5 is complete. ■

5

The Case of (2,3)-Manifolds

In this chapter we finalize our classification of connected non-homogeneous hyperbolic manifolds of dimension $n \geq 2$ with $d(M) = n^2 - 1$ by considering the case $n = 2$. We then have $d(M) = 3$, and, as we stated in Sect. 1.4, for brevity we call connected 2-dimensional hyperbolic manifolds with 3-dimensional automorphism group (2,3)-manifolds. Proposition 4.2 gives that if M is a (2,3)-manifold, then for every $p \in M$ the orbit $O(p)$ has (real) codimension 1 or 2 in M, and in the latter case $O(p)$ is either a complex curve or a totally real submanifold. We start by making the following remark.

Remark 5.1. If no codimension 1 orbits are present in a (2,3)-manifold M, then, arguing as in Sect. 4.4, one can prove that M is holomorphically equivalent to $\Delta \times S$, where S is a hyperbolic Riemann surface with $d(S) = 0$.

Thus, from now on we only consider (2,3)-manifolds with codimension 1 orbits. Clearly, every codimension 1 orbit is either strongly pseudoconvex or Levi-flat.

The chapter is organized as follows. In Sect. 5.1 we give a large number of examples of (2,3)-manifolds. It will be shown in later sections that in fact these examples form a complete classification of (2,3)-manifolds with codimension 1 orbits.

In Sect. 5.2 we deal with the case when every orbit is strongly pseudoconvex and classify all (2,3)-manifolds with this property in Theorem 5.2. An important ingredient in the proof of Theorem 5.2 is E. Cartan's classification of 3-dimensional homogeneous strongly pseudoconvex CR-manifolds (see [CaE1]), together with the explicit determination of all covers of the non simply-connected hypersurfaces on Cartan's list (see [I4]). The explicit realizations of the covers are important for our arguments throughout the chapter, especially for those in the proof of Theorem 5.7 in Sect. 5.4. Another ingredient in the proof of Theorem 5.2 is an orbit gluing procedure similar to that used in Sect. 3.4 that allows us to join strongly pseudoconvex orbits together to form (2,3)-manifolds.

Studying situations when Levi-flat and codimension 2 orbits can occur is perhaps the most interesting part of the chapter. In Sect. 5.3 we deal with Levi-flat orbits. According to (ii) of Proposition 4.2, every such orbit is foliated by complex manifolds equivalent to Δ. We describe Levi-flat orbits together with all possible actions of $G(M)$ in Proposition 5.4 and use this description to classify in Theorem 5.6 all (2,3)-manifolds for which every orbit has codimension 1 and at least one orbit is Levi-flat.

Finally, in Sect. 5.4 we allow codimension 2 orbits to be present in the manifold. Every complex curve orbit is equivalent to Δ (see (iii) of Proposition 4.2), whereas no initial description of totally real orbits is available. As before, the properness of the $G(M)$-action implies that there are at most two codimension 2 orbits in M, and in the proof of Theorem 5.7 we investigate how one or two such orbits can be added to the previously obtained manifolds. This is done by studying complex curves invariant under the actions of the isotropy subgroups of points lying in codimension 2 orbits.

To summarize, our classification of (2,3)-manifolds is contained in Remark 5.1 and Theorems 5.2, 5.6, 5.7.

5.1 Examples of (2,3)-Manifolds

In this section we give a large number of examples of (2,3)-manifolds. It will be shown in the forthcoming sections that these examples (upon excluding equivalent manifolds) give a complete classification of (2,3)-manifolds with codimension 1 orbits.

(1) In this example strongly pseudoconvex and Levi-flat orbits occur.

(a) Fix $\alpha \in \mathbb{R}$, $\alpha \neq 0, 1$, and choose $0 \leq s < t \leq \infty$ with either $s > 0$ or $t < \infty$. Define

$$R_{\alpha,s,t} := \left\{(z,w) \in \mathbb{C}^2 : s\,(\mathrm{Re}\, z)^{\alpha} < \mathrm{Re}\, w < t\,(\mathrm{Re}\, z)^{\alpha}\,,\ \mathrm{Re}\, z > 0\right\}. \tag{5.1}$$

The group $G(R_{\alpha,s,t}) = \mathrm{Aut}(R_{\alpha,s,t})$ consists of all maps

$$\begin{aligned} z &\mapsto \lambda z + i\beta, \\ w &\mapsto \lambda^{\alpha} w + i\gamma, \end{aligned} \tag{5.2}$$

where $\lambda > 0$ and $\beta, \gamma \in \mathbb{R}$. The $G(R_{\alpha,s,t})$-orbits are the following pairwise CR-equivalent strongly pseudoconvex hypersurfaces:

$$O_{\nu}^{R_{\alpha}} := \left\{(z,w) \in \mathbb{C}^2 : \mathrm{Re}\, w = \nu\,(\mathrm{Re}\, z)^{\alpha}\,,\ \mathrm{Re}\, z > 0\right\}, \quad s < \nu < t,$$

and we set

$$\tau_{\alpha} := O_1^{R_{\alpha}}. \tag{5.3}$$

For every $\alpha \in \mathbb{R}$ we denote the group of maps of the form (5.2) by G_{α}.

(b) If in the definition of $R_{\alpha,s,t}$ we let $-\infty \le s < 0 < t \le \infty$, where at least one of s, t is finite, we again obtain a hyperbolic domain whose automorphism group coincides with G_α, unless $\alpha = 1/2$ and $t = -s$ (observe that $R_{1/2,s,-s}$ is equivalent to $\mathbb{B}^2$). In such domains, in addition to the strongly pseudoconvex orbits $O_\nu^{R_\alpha}$ for suitable values of ν (which are allowed to be negative), there is the following unique Levi-flat orbit:

$$\mathcal{O}_1 := \left\{(z,w) \in \mathbb{C}^2 : \operatorname{Re} z > 0,\ \operatorname{Re} w = 0\right\}. \tag{5.4}$$

(c) For $\alpha > 0$, $\alpha \ne 1$, $-\infty < s < 0 < t < \infty$ define

$$\hat{R}_{\alpha,s,t} := R_{\alpha,s,\infty} \cup \left\{(z,w) \in \mathbb{C}^2 : \operatorname{Re} w > t\,(-\operatorname{Re} z)^\alpha,\ \operatorname{Re} z < 0\right\} \cup \hat{\mathcal{O}}_1,$$

where

$$\hat{\mathcal{O}}_1 := \left\{(z,w) \in \mathbb{C}^2 : \operatorname{Re} z = 0,\ \operatorname{Re} w > 0\right\}. \tag{5.5}$$

The group $G(\hat{R}_{\alpha,s,t}) = \operatorname{Aut}(\hat{R}_{\alpha,s,t})$ coincides with G_α, and, in addition to strongly pseudoconvex orbits CR-equivalent to τ_α, the Levi-flat hypersurfaces $\mathcal{O}_1$ and $\hat{\mathcal{O}}_1$ are also G_α-orbits in $\hat{R}_{\alpha,s,t}$.

(2) In this example strongly pseudoconvex orbits and a single Levi-flat orbit arise.

(a) For $0 \le s < t \le \infty$ with either $s > 0$ or $t < \infty$ define

$$U_{s,t} := \Big\{(z,w) \in \mathbb{C}^2 : \operatorname{Re} w \cdot \ln(s\operatorname{Re} w) < \operatorname{Re} z < \operatorname{Re} w \cdot \ln(t\operatorname{Re} w),\ \operatorname{Re} w > 0\Big\}. \tag{5.6}$$

The group $G(U_{s,t}) = \operatorname{Aut}(U_{s,t})$ consists of all maps

$$\begin{aligned} z &\mapsto \lambda z + (\lambda \ln \lambda) w + i\beta,\\ w &\mapsto \lambda w + i\gamma, \end{aligned} \tag{5.7}$$

where $\lambda > 0$ and $\beta, \gamma \in \mathbb{R}$. The $G(U_{s,t})$-orbits are the following pairwise CR-equivalent strongly pseudoconvex hypersurfaces:

$$O_\nu^U := \left\{(z,w) \in \mathbb{C}^2 : \operatorname{Re} z = \operatorname{Re} w \cdot \ln(\nu\operatorname{Re} w),\ \operatorname{Re} w > 0\right\}, \quad s < \nu < t,$$

and we set

$$\xi := O_1^U. \tag{5.8}$$

We denote the group of all maps of the form (5.7) by $\mathfrak{G}$.

(b) For $-\infty < t < 0 < s < \infty$ define

$$\hat{U}_{s,t} = U_{s,\infty} \cup \left\{(z,w) \in \mathbb{C}^2 : \operatorname{Re} z > \operatorname{Re} w \cdot \ln(t\operatorname{Re} w),\ \operatorname{Re} w < 0\right\} \cup \mathcal{O}_1.$$

The group $G(\hat{U}_{s,t}) = \mathrm{Aut}(\hat{U}_{s,t})$ coincides with $\mathfrak{G}$, and, in addition to strongly pseudoconvex orbits CR-equivalent to ξ, the Levi-flat hypersurface $\mathcal{O}_1$ is also a $\mathfrak{G}$-orbit in $\hat{U}_{s,t}$.

(3) In this example strongly pseudoconvex orbits and a totally real orbit occur.

(a) For $0 \le s < t < \infty$ define

$$\mathfrak{S}_{s,t} := \left\{(z,w) \in \mathbb{C}^2 : s < (\mathrm{Re}\, z)^2 + (\mathrm{Re}\, w)^2 < t\right\}. \tag{5.9}$$

The group $G(\mathfrak{S}_{s,t})$ consists of all maps of the form

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto A \begin{pmatrix} z \\ w \end{pmatrix} + i \begin{pmatrix} \beta \\ \gamma \end{pmatrix}, \tag{5.10}$$

where $A \in SO_2(\mathbb{R})$ and $\beta, \gamma \in \mathbb{R}$. The $G(\mathfrak{S}_{s,t})$-orbits are the following pairwise CR-equivalent strongly pseudoconvex hypersurfaces:

$$O_\nu^{\mathfrak{S}} := \left\{(z,w) \in \mathbb{C}^2 : (\mathrm{Re}\, z)^2 + (\mathrm{Re}\, w)^2 = \nu\right\}, \quad s < \nu < t,$$

and we set

$$\chi := O_1^{\mathfrak{S}}. \tag{5.11}$$

We denote the group of all maps of the form (5.10) by $\mathcal{R}_\chi$.

(b) For $0 < t < \infty$ set

$$\mathfrak{S}_t := \left\{(z,w) \in \mathbb{C}^2 : (\mathrm{Re}\, z)^2 + (\mathrm{Re}\, w)^2 < t\right\}. \tag{5.12}$$

The group $G(\mathfrak{S}_t)$ coincides with $\mathcal{R}_\chi$, and, apart from strongly pseudoconvex orbits CR-equivalent to χ, its action on $\mathfrak{S}_t$ has the totally real orbit

$$\mathcal{O}_2 := \left\{(z,w) \in \mathbb{C}^2 : \mathrm{Re}\, z = 0,\ \mathrm{Re}\, w = 0\right\}. \tag{5.13}$$

(4) In this example we explicitly describe all covers of the domains $\mathfrak{S}_{s,t}$ and hypersurface χ introduced in **(3)** (for more details see [I4]). Only strongly pseudoconvex orbits occur here.

Let $\Phi_\chi^{(\infty)} : \mathbb{C}^2 \to \mathbb{C}^2$ be the following map:

$$\begin{aligned} z &\mapsto \exp(\mathrm{Re}\, z)\cos(\mathrm{Im}\, z) + i\mathrm{Re}\, w, \\ w &\mapsto \exp(\mathrm{Re}\, z)\sin(\mathrm{Im}\, z) + i\mathrm{Im}\, w. \end{aligned}$$

It is easy to see that $\Phi_\chi^{(\infty)}$ is an infinitely-sheeted covering map onto $\mathbb{C}^2 \setminus \{\mathrm{Re}\, z = 0,\ \mathrm{Re}\, w = 0\}$. Introduce on the domain of $\Phi_\chi^{(\infty)}$ the complex structure defined by the condition that the map $\Phi_\chi^{(\infty)}$ is holomorphic (the *pull-back complex structure under* $\Phi_\chi^{(\infty)}$), and denote the resulting manifold by $M_\chi^{(\infty)}$. Next, for an integer $n \geq 2$, consider the map $\Phi_\chi^{(n)}$ from $\mathbb{C}^2 \setminus \{\mathrm{Re}\, z = 0,\ \mathrm{Re}\, w = 0\}$ onto itself defined as follows:

$$\begin{aligned} z &\mapsto \mathrm{Re}\Big((\mathrm{Re}\, z + i\mathrm{Re}\, w)^n\Big) + i\mathrm{Im}\, z,\\ w &\mapsto \mathrm{Im}\Big((\mathrm{Re}\, z + i\mathrm{Re}\, w)^n\Big) + i\mathrm{Im}\, w. \end{aligned} \tag{5.14}$$

Denote by $M_\chi^{(n)}$ the domain of $\Phi_\chi^{(n)}$ with the pull-back complex structure under $\Phi_\chi^{(n)}$.

For $0 \leq s < t < \infty$, $n \geq 2$ define

$$\begin{aligned} \mathfrak{S}_{s,t}^{(n)} &:= \left\{(z,w) \in M_\chi^{(n)} : s^{1/n} < (\mathrm{Re}\, z)^2 + (\mathrm{Re}\, w)^2 < t^{1/n}\right\},\\ \mathfrak{S}_{s,t}^{(\infty)} &:= \left\{(z,w) \in M_\chi^{(\infty)} : (\ln s)/2 < \mathrm{Re}\, z < (\ln t)/2\right\}. \end{aligned} \tag{5.15}$$

The domains $\mathfrak{S}_{s,t}^{(n)}$ and $\mathfrak{S}_{s,t}^{(\infty)}$ are respectively an n- and infinite-sheeted cover of the domain $\mathfrak{S}_{s,t}$. The group $G\left(\mathfrak{S}_{s,t}^{(n)}\right)$ for $n \geq 2$ consists of all maps

$$\begin{aligned} z \mapsto\ & \cos\psi \cdot \mathrm{Re}\, z + \sin\psi \cdot \mathrm{Re}\, w +\\ & i\Big(\cos(n\psi)\cdot \mathrm{Im}\, z + \sin(n\psi)\cdot \mathrm{Im}\, w + \beta\Big),\\ w \mapsto\ & -\sin\psi \cdot \mathrm{Re}\, z + \cos\psi \cdot \mathrm{Re}\, w +\\ & i\Big(-\sin(n\psi)\cdot \mathrm{Im}\, z + \cos(n\psi)\cdot \mathrm{Im}\, w + \gamma\Big), \end{aligned} \tag{5.16}$$

where $\psi, \beta, \gamma \in \mathbb{R}$. The $G\left(\mathfrak{S}_{s,t}^{(n)}\right)$-orbits are the following pairwise CR-equivalent strongly pseudoconvex hypersurfaces:

$$O_\nu^{\mathfrak{S}^{(n)}} := \left\{(z,w) \in M_\chi^{(n)} : (\mathrm{Re}\, z)^2 + (\mathrm{Re}\, w)^2 = \nu\right\}, \quad s^{1/n} < \nu < t^{1/n},$$

and we set

$$\chi^{(n)} := O_1^{\mathfrak{S}^{(n)}} \tag{5.17}$$

(this hypersurface is an n-sheeted cover of χ).

The group $G\left(\mathfrak{S}_{s,t}^{(\infty)}\right)$ consists of all maps

$$\begin{aligned} z &\mapsto\ z + i\beta,\\ w &\mapsto e^{i\beta} w + a, \end{aligned} \tag{5.18}$$

where $\beta \in \mathbb{R}$, $a \in \mathbb{C}$. The $G\left(\mathfrak{S}_{s,t}^{(\infty)}\right)$-orbits are the following pairwise CR-equivalent strongly pseudoconvex hypersurfaces:

$$O_\nu^{\mathfrak{S}^{(\infty)}} := \left\{(z,w) \in M_\chi^{(\infty)} : \operatorname{Re} z = \nu\right\}, \quad (\ln s)/2 < \nu < (\ln t)/2,$$

and we set

$$\chi^{(\infty)} := O_0^{\mathfrak{S}^{(\infty)}} \tag{5.19}$$

(this hypersurface is an infinitely-sheeted cover of χ).

(5) As in the preceding example, only strongly pseudoconvex orbits occur here.

Fix $\alpha > 0$ and for $0 < t < \infty$, $e^{-2\pi\alpha}t < s < t$ consider the tube domain

$$V_{\alpha,s,t} := \left\{(z,w) \in \mathbb{C}^2 : se^{\alpha\phi} < r < te^{\alpha\phi}\right\}, \tag{5.20}$$

where (r,ϕ) denote the polar coordinates in the $(\operatorname{Re} z, \operatorname{Re} w)$-plane with ϕ varying from $-\infty$ to ∞ (thus, the boundary of $V_{\alpha,t,s} \cap \mathbb{R}^2$ consists of two spirals accumulating to the origin and infinity). The group $G(V_{\alpha,s,t}) = \operatorname{Aut}(V_{\alpha,s,t})$ consists of all maps of the form

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto e^{\alpha\psi} \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} + i \begin{pmatrix} \beta \\ \gamma \end{pmatrix}, \tag{5.21}$$

where $\psi, \beta, \gamma \in \mathbb{R}$. The $G(V_{\alpha,s,t})$-orbits are the following pairwise CR-equivalent strongly pseudoconvex hypersurfaces:

$$O_\nu^{V_\alpha} := \left\{(z,w) \in \mathbb{C}^2 : r = \nu e^{\alpha\phi}\right\}, \quad s < \nu < t,$$

and we set

$$\rho_\alpha := O_1^{V_\alpha}. \tag{5.22}$$

(6) In this example strongly pseudoconvex orbits and a totally real orbit arise.

(a) For $1 \le s < t < \infty$ define

$$E_{s,t} := \Big\{(\zeta : z : w) \in \mathbb{CP}^2 : s|\zeta^2 + z^2 + w^2| < |\zeta|^2 + |z|^2 + |w|^2 < t|\zeta^2 + z^2 + w^2|\Big\}. \tag{5.23}$$

The group $G(E_{s,t}) = \operatorname{Aut}(E_{s,t})$ is given by

$$\begin{pmatrix} \zeta \\ z \\ w \end{pmatrix} \mapsto A \begin{pmatrix} \zeta \\ z \\ w \end{pmatrix}, \tag{5.24}$$

where $A \in SO_3(\mathbb{R})$. The orbits of the action of the group $G(E_{s,t})$ on $E_{s,t}$ are the following pairwise CR non-equivalent strongly pseudoconvex hypersurfaces:

$$\mu_\alpha := \left\{(\zeta : z : w) \in \mathbb{CP}^2 : |\zeta|^2 + |z|^2 + |w|^2 = \alpha|\zeta^2 + z^2 + w^2|\right\}, \quad s < \alpha < t. \tag{5.25}$$

We denote the group of all maps of the form (5.24) by $\mathcal{R}_\mu$.

(b) For $1 < t < \infty$ define

$$E_t := \Big\{(\zeta : z : w) \in \mathbb{CP}^2 : |\zeta|^2 + |z|^2 + |w|^2 < t|\zeta^2 + z^2 + w^2|\Big\}. \tag{5.26}$$

The group $G(E_t)$ coincides with $\mathcal{R}_\mu$, and its action on E_t has, apart from strongly pseudoconvex orbits, the following totally real orbit:

$$\mathcal{O}_3 := \mathbb{RP}^2 \subset \mathbb{CP}^2. \tag{5.27}$$

(7) Here we explicitly describe all covers of the domains $E_{s,t}$ and hypersurfaces μ_α introduced in **(6)** (for more details see [I4]). As we will see below, to one of the covers of $E_{1,t}$ a totally real orbit can be attached.

(a) Let $\mathcal{Q}_+$ be the variety in $\mathbb{C}^3$ given by

$$z_1^2 + z_2^2 + z_3^2 = 1. \tag{5.28}$$

Consider the map $\Phi_\mu : \mathbb{C}^2 \setminus \{0\} \to \mathcal{Q}_+$ defined by the formulas

$$\begin{aligned} z_1 &= -i(z^2 + w^2) + i\frac{z\overline{w} - w\overline{z}}{|z|^2 + |w|^2}, \\ z_2 &= z^2 - w^2 - \frac{z\overline{w} + w\overline{z}}{|z|^2 + |w|^2}, \\ z_3 &= 2zw + \frac{|z|^2 - |w|^2}{|z|^2 + |w|^2}. \end{aligned} \tag{5.29}$$

This map was introduced in [Ros]. It is straightforward to verify that Φ_μ is a 2-to-1 covering map onto $\mathcal{Q}_+ \setminus \mathbb{R}^3$. We now equip the domain of Φ_μ with the pull-back complex structure under Φ_μ and denote the resulting complex manifold by $M_\mu^{(4)}$.

For $1 \le s < t < \infty$ define

$$\begin{aligned} E_{s,t}^{(2)} &:= \Big\{(z_1, z_2, z_3) \in \mathbb{C}^3 : s < |z_1|^2 + |z_2|^2 + |z_3|^2 < t\Big\} \cap \mathcal{Q}_+, \\ E_{s,t}^{(4)} &:= \Big\{(z, w) \in M_\mu^{(4)} : \sqrt{(s-1)/2} < |z|^2 + |w|^2 < \sqrt{(t-1)/2}\Big\}. \end{aligned} \tag{5.30}$$

These domains are respectively a 2- and 4-sheeted cover of the domain $E_{s,t}$, where $E_{s,t}^{(2)}$ covers $E_{s,t}$ by means of the map $\Psi_\mu : (z_1, z_2, z_3) \mapsto (z_1 : z_2 : z_3)$ and $E_{s,t}^{(4)}$ covers $E_{s,t}$ by means of the composition $\Psi_\mu \circ \Phi_\mu$.

The group $G\left(E^{(2)}_{s,t}\right)$ consists of all maps

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \mapsto A \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}, \tag{5.31}$$

where $A \in SO_3(\mathbb{R})$. The $G\left(E^{(2)}_{s,t}\right)$-orbits are the following pairwise CR non-equivalent strongly pseudoconvex hypersurfaces:

$$\mu^{(2)}_\alpha := \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 + |z_3|^2 = \alpha\right\} \cap \mathcal{Q}_+,\ s < \alpha < t \tag{5.32}$$

(note that $\mu^{(2)}_\alpha$ is a 2-sheeted cover of μ_α). We denote the group of all maps of the form (5.31) by $\mathcal{R}^{(2)}_\mu$. This group is clearly isomorphic to $\mathcal{R}_\mu$.

The group $G\left(E^{(4)}_{s,t}\right)$ consists of all maps

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto A \begin{pmatrix} z \\ w \end{pmatrix}, \tag{5.33}$$

where $A \in SU_2$. The $G\left(E^{(4)}_{s,t}\right)$-orbits are the following pairwise CR non-equivalent strongly pseudoconvex hypersurfaces:

$$\mu^{(4)}_\alpha := \left\{(z, w) \in M^{(4)}_\mu : |z|^2 + |w|^2 = \sqrt{(\alpha - 1)/2}\right\}, \quad s < \alpha < t \tag{5.34}$$

(note that $\mu^{(4)}_\alpha$ is a 4-sheeted cover of μ_α).

(b) For $1 < t < \infty$ define

$$E^{(2)}_t := \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 + |z_3|^2 < t\right\} \cap \mathcal{Q}_+. \tag{5.35}$$

The group $G\left(E^{(2)}_t\right)$ coincides with $\mathcal{R}^{(2)}_\mu$, and, apart from strongly pseudoconvex orbits, its action on $E^{(2)}_t$ has the totally real orbit

$$\mathcal{O}_4 := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\right\} = \mathcal{Q}_+ \cap \mathbb{R}^3. \tag{5.36}$$

(8) In this example strongly pseudoconvex orbits and a totally real orbit arise.

(a) For $-1 \le s < t \le 1$ define

$$\Omega_{s,t} := \Big\{(z, w) \in \mathbb{C}^2 : s|z^2 + w^2 - 1| < |z|^2 + |w|^2 - 1 < t|z^2 + w^2 - 1|\Big\}. \tag{5.37}$$

The group $G(\Omega_{s,t})$ consists of all maps

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \frac{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}}{c_1 z + c_2 w + d}, \tag{5.38}$$

where

$$Q := \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ c_1 & c_2 & d \end{pmatrix} \in SO_{2,1}(\mathbb{R})^0. \tag{5.39}$$

The orbits of $G(\Omega_{s,t})$ on $\Omega_{s,t}$ are the following pairwise CR non-equivalent strongly pseudoconvex hypersurfaces:

$$\begin{aligned} \nu_\alpha := &\left\{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 - 1 = \alpha|z^2 + w^2 - 1|\right\} \setminus \\ &\left\{(x,u) \in \mathbb{R}^2 : x^2 + u^2 = 1\right\}, \quad s < \alpha < t. \end{aligned} \tag{5.40}$$

We denote the group of all maps of the form (5.38) by $\mathcal{R}_\nu$.

(b) For $-1 < t \leq 1$ define

$$\Omega_t := \left\{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 - 1 < t|z^2 + w^2 - 1|\right\}. \tag{5.41}$$

The group $G(\Omega_t)$ for $t < 1$ coincides with $\mathcal{R}_\nu$, and its action on Ω_t, apart from strongly pseudoconvex orbits, has the totally real orbit

$$\mathcal{O}_5 := \left\{(x,u) \in \mathbb{R}^2 : x^2 + u^2 < 1\right\} \subset \mathbb{C}^2. \tag{5.42}$$

We note that Ω_1 is holomorphically equivalent to Δ^2 (see **(11)(c)** below); hence it has a 6-dimensional automorphism group and therefore will be excluded from our considerations.

(9) In this example strongly pseudoconvex orbits and a complex curve orbit occur.

(a) For $1 \leq s < t \leq \infty$ define

$$\begin{aligned} D_{s,t} := \Big\{(z,w) \in \mathbb{C}^2 : s|1 + z^2 - w^2| < 1 + |z|^2 - |w|^2 < \\ t|1 + z^2 - w^2|, \operatorname{Im}(z(1+\overline{w})) > 0\Big\}, \end{aligned} \tag{5.43}$$

where $D_{s,\infty}$ is assumed not to include the complex curve

$$\mathcal{O} := \left\{(z,w) \in \mathbb{C}^2 : 1 + z^2 - w^2 = 0, \operatorname{Im}(z(1+\overline{w})) > 0\right\}. \tag{5.44}$$

For every matrix $Q \in SO_{2,1}(\mathbb{R})^0$ as in (5.39) consider the map

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \frac{\begin{pmatrix} a_{22} & b_2 \\ c_2 & d \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} + \begin{pmatrix} a_{21} \\ c_1 \end{pmatrix}}{a_{12}z + b_1 w + a_{11}}. \tag{5.45}$$

The group $G(D_{s,t}) = \text{Aut}(D_{s,t})$ consists of all such maps. The orbits of $G(D_{s,t})$ on $D_{s,t}$ are the following pairwise CR non-equivalent strongly pseudoconvex hypersurfaces:

$$\begin{aligned} \eta_\alpha := \Big\{ &(z, w) \in \mathbb{C}^2 : 1 + |z|^2 - |w|^2 = \alpha|1 + z^2 - w^2|, \\ &\text{Im}(z(1 + \overline{w})) > 0 \Big\}, \quad s < \alpha < t. \end{aligned} \tag{5.46}$$

We denote the group of all maps of the form (5.45) by $\mathcal{R}_\eta$ (note that $\mathcal{R}_\eta$ is isomorphic to $\mathcal{R}_\nu$).

(b) For $1 \leq s < \infty$ define

$$\begin{aligned} D_s := \Big\{ (z, w) \in \mathbb{C}^2 : 1 + |z|^2 - |w|^2 > s|1 + z^2 - w^2|, \\ \text{Im}\,(z(1 + \overline{w})) > 0 \Big\}, \end{aligned} \tag{5.47}$$

The group $G(D_s) = \text{Aut}(D_s)$ coincides with $\mathcal{R}_\eta$. Apart from strongly pseudoconvex orbits, its action on D_s has the complex curve orbit $\mathcal{O}$.

(10) In this example we explicitly describe all covers of the domains $\Omega_{s,t}$, $D_{s,t}$ and the hypersurfaces ν_α, η_α introduced in **(8)** and **(9)** (for more details see [I4]). Only strongly pseudoconvex orbits occur here.

Denote by $(z_0 : z_1 : z_2 : z_3)$ homogeneous coordinates in $\mathbb{CP}^3$; we think of the hypersurface $\{z_0 = 0\}$ as the infinity. Let $\mathcal{Q}_-$ be the variety in $\mathbb{CP}^3$ given by

$$z_1^2 + z_2^2 - z_3^2 = z_0^2. \tag{5.48}$$

Next, let $(\zeta : z : w)$ be homogeneous coordinates in $\mathbb{CP}^2$ (where we think of the hypersurface $\{\zeta = 0\}$ as the infinity), and let

$$\Sigma := \left\{ (\zeta : z : w) \in \mathbb{CP}^2 : |w| < |z| \right\}. \tag{5.49}$$

For every integer $n \geq 2$ consider the map $\Phi^{(n)}$ from Σ to $\mathcal{Q}_-$ defined as follows:

$$\begin{aligned} z_0 &= \zeta^n, \\ z_1 &= -i(z^n + z^{n-2}w^2) - i\frac{z\overline{w} + w\overline{z}}{|z|^2 - |w|^2}\zeta^n, \\ z_2 &= z^n - z^{n-2}w^2 + \frac{z\overline{w} - w\overline{z}}{|z|^2 - |w|^2}\zeta^n, \\ z_3 &= -2iz^{n-1}w - i\frac{|z|^2 + |w|^2}{|z|^2 - |w|^2}\zeta^n. \end{aligned} \tag{5.50}$$

The above maps were introduced in [I4] and are analogous to the map Φ_μ defined in (5.29). Further, set

$$\begin{aligned} \mathcal{A}_\nu^{(n)} &:= \left\{(z,w)\in\mathbb{C}^2 : 0<|z|^n-|z|^{n-2}|w|^2<1\right\},\\ \mathcal{A}_\eta^{(n)} &:= \left\{(z,w)\in\mathbb{C}^2 : |z|^n-|z|^{n-2}|w|^2>1\right\} \end{aligned} \tag{5.51}$$

(both domains lie in the finite part of $\mathbb{CP}^2$ given by $\zeta = 1$). Clearly, $\mathcal{A}_\nu^{(n)}$, $\mathcal{A}_\eta^{(n)} \subset \Sigma$ for all $n \geq 2$. Let $\Phi_\nu^{(n)}$ and $\Phi_\eta^{(n)}$ be the restrictions of $\Phi^{(n)}$ to $\mathcal{A}_\nu^{(n)}$ and $\mathcal{A}_\eta^{(n)}$, respectively. It is straightforward to observe that $\Phi_\nu^{(n)}$ and $\Phi_\eta^{(n)}$ are n-to-1 covering maps onto

$$\begin{aligned} \mathcal{A}_\nu := \Big\{(z_1,z_2,z_3)\in\mathbb{C}^3 : -1<|z_1|^2+|z_2|^2-|z_3|^2<1,\\ \operatorname{Im} z_3<0\Big\}\cap\mathcal{Q}_- \end{aligned} \tag{5.52}$$

and

$$\begin{aligned} \mathcal{A}_\eta := \Big\{(z_1,z_2,z_3)\in\mathbb{C}^3 : |z_1|^2+|z_2|^2-|z_3|^2>1,\\ \operatorname{Im}(z_2(\overline{z_1}+\overline{z_3}))>0\Big\}\cap\mathcal{Q}_-, \end{aligned} \tag{5.53}$$

respectively (both domains lie in the finite part of $\mathbb{CP}^3$ given by $z_0 = 1$). We now introduce on $\mathcal{A}_\nu^{(n)}$, $\mathcal{A}_\eta^{(n)}$ the pull-back complex structures under the maps $\Phi_\nu^{(n)}$, $\Phi_\eta^{(n)}$, respectively, and denote the resulting complex manifolds by $M_\nu^{(n)}$, $M_\eta^{(n)}$.

Further, let $\Lambda : \mathbb{C}\times\Delta \to \Sigma\cap\{\zeta=1\}$ be the following covering map:

$$\begin{aligned} z &\mapsto e^z,\\ w &\mapsto e^z w, \end{aligned} \tag{5.54}$$

where $z \in \mathbb{C}$, $w \in \Delta$. Define

$$\begin{aligned} U_\nu &:= \left\{(z,w)\in\mathbb{C}^2 : |w|<1,\ \exp(2\operatorname{Re} z)(1-|w|^2)<1\right\},\\ U_\eta &:= \left\{(z,w)\in\mathbb{C}^2 : |w|<1,\ \exp(2\operatorname{Re} z)(1-|w|^2)>1\right\}. \end{aligned}$$

Denote by Λ_ν, Λ_η the restrictions of Λ to U_ν, U_η, respectively. Clearly, U_ν covers $M_\nu^{(2)}$ by means of Λ_ν, and U_η covers $M_\eta^{(2)}$ by means of Λ_η. We now introduce on U_ν, U_η the pull-back complex structures under the maps Λ_ν, Λ_η, respectively, and denote the resulting complex manifolds by $M_\nu^{(\infty)}$, $M_\eta^{(\infty)}$.

For $-1 \leq s < t \leq 1$, $n \geq 2$ we now define

$$\begin{aligned} \Omega_{s,t}^{(n)} &:= \Big\{(z,w)\in M_\nu^{(n)} : \sqrt{(s+1)/2}<|z|^n-|z|^{n-2}|w|^2<\\ &\qquad \sqrt{(t+1)/2}\Big\},\\ \Omega_{s,t}^{(\infty)} &:= \Big\{(z,w)\in M_\nu^{(\infty)} : \sqrt{(s+1)/2}<\exp(2\operatorname{Re} z)(1-|w|^2)<\\ &\qquad \sqrt{(t+1)/2}\Big\}. \end{aligned} \tag{5.55}$$

The domain $\Omega_{s,t}^{(n)}$, $n \geq 2$, is an n-sheeted cover of the domain $\Omega_{s,t}$ introduced in **(8)** and the domain $\Omega_{s,t}^{(\infty)}$ is its infinitely-sheeted cover. The domain $\Omega_{s,t}^{(n)}$ covers $\Omega_{s,t}$ by means of the composition $\Psi_\nu \circ \Phi_\nu^{(n)}$, where Ψ_ν is the following 1-to-1 map from $\mathcal{A}_\nu$ to $\mathbb{C}^2$: $(z_1, z_2, z_3) \mapsto (z_1/z_3, z_2/z_3)$; the domain $\Omega_{s,t}^{(\infty)}$ covers $\Omega_{s,t}$ by means of the composition $\Psi_\nu \circ \Phi_\nu^{(2)} \circ \Lambda_\nu$.

The group $G\left(\Omega_{s,t}^{(n)}\right)$ consists of all maps of the form

$$\begin{aligned} z &\mapsto z \sqrt[n]{(a + b\, w/z)^2}, \\ w &\mapsto z \frac{\bar{b} + \bar{a}\, w/z}{a + b\, w/z} \sqrt[n]{(a + b\, w/z)^2}, \end{aligned} \tag{5.56}$$

where $|a|^2 - |b|^2 = 1$. The $G\left(\Omega_{s,t}^{(n)}\right)$-orbits are the following pairwise CR non-equivalent strongly pseudoconvex hypersurfaces:

$$\nu_\alpha^{(n)} := \left\{ (z, w) \in M_\nu^{(n)} : |z|^n - |z|^{n-2}|w|^2 = \sqrt{(\alpha+1)/2} \right\}, \quad s < \alpha < t \tag{5.57}$$

(note that $\nu_\alpha^{(n)}$ is an n-sheeted cover of ν_α). We denote the group of all maps of the form (5.56) by $\mathcal{R}^{(n)}$.

The group $G\left(\Omega_{s,t}^{(\infty)}\right)$ consists of all maps of the form

$$\begin{aligned} z &\mapsto z + \ln(a + bw), \\ w &\mapsto \frac{\bar{b} + \bar{a}w}{a + bw}, \end{aligned} \tag{5.58}$$

where $|a|^2 - |b|^2 = 1$. The $G\left(\Omega_{s,t}^{(\infty)}\right)$-orbits are the following pairwise CR non-equivalent strongly pseudoconvex hypersurfaces:

$$\nu_\alpha^{(\infty)} := \left\{ (z, w) \in M_\nu^{(\infty)} : \exp\left(2\mathrm{Re}\, z\right)\left(1 - |w|^2\right) = \sqrt{(\alpha+1)/2} \right\}, \quad s < \alpha < t \tag{5.59}$$

(note that $\nu_\alpha^{(\infty)}$ is an infinitely-sheeted cover of ν_α). We denote the group of all maps of the form (5.58) by $\mathcal{R}^{(\infty)}$.

Next, for $1 \leq s < t \leq \infty$, $n \geq 2$ we define

$$
\begin{aligned}
D_{s,t}^{(2)} &:= \Big\{(z_1,z_2,z_3)\in\mathbb{C}^3 : s<|z_1|^2+|z_2|^2-|z_3|^2<t,\\
&\qquad\qquad \operatorname{Im}(z_2(\overline{z_1}+\overline{z_3}))>0\Big\}\cap\mathcal{Q}_-,\\
D_{s,t}^{(2n)} &:= \Big\{(z,w)\in M_\eta^{(n)} : \sqrt{(s+1)/2}<|z|^n-|z|^{n-2}|w|^2<\\
&\qquad\qquad \sqrt{(t+1)/2}\Big\},\\
D_{s,t}^{(\infty)} &:= \Big\{(z,w)\in M_\eta^{(\infty)} : \sqrt{(s+1)/2}<\\
&\qquad\qquad \exp\left(2\mathrm{Re}\, z\right)\left(1-|w|^2\right)<\sqrt{(t+1)/2}\Big\}.
\end{aligned}
\tag{5.60}
$$

The domain $D_{s,t}^{(2n)}$, $n\ge 1$, is a $2n$-sheeted cover of the domain $D_{s,t}$ introduced in **(9)** and the domain $D_{s,t}^{(\infty)}$ is its infinitely-sheeted cover. The domain $D_{s,t}^{(2)}$ covers $D_{s,t}$ by means of the map Ψ_η, which is the following 2-to-1 map from $\mathcal{A}_\eta$ to $\mathbb{C}^2 : (z_1,z_2,z_3)\mapsto(z_2/z_1, z_3/z_1)$; the domain $D_{s,t}^{(2n)}$ for $n\ge 2$ covers $D_{s,t}$ by means of the composition $\Psi_\eta\circ\Phi_\eta^{(n)}$; the domain $D_{s,t}^{(\infty)}$ covers $D_{s,t}$ by means of the composition $\Psi_\eta\circ\Phi_\eta^{(2)}\circ\Lambda_\eta$.

To obtain an n-sheeted cover of $D_{s,t}$ for odd $n\ge 3$, we take the quotient of the domain $D_{s,t}^{(4n)}$ by the action of the cyclic group of four elements generated by the following automorphism of $M_\eta^{(2n)}$:

$$
\begin{aligned}
z &\mapsto iz^2\overline{z}\sqrt[n]{\frac{1-|w|^2/|z|^2+z^{-2n}\overline{w}/\overline{z}}{\sqrt{|z|^{4n}(1-|w|^2/|z|^2)^2-1}}},\\
w &\mapsto i\frac{1+z^{2n-1}w(1-|w|^2/|z|^2)}{\overline{w}/\overline{z}+z^{2n}(1-|w|^2/|z|^2)}\times\\
&\qquad z^2\overline{z}\sqrt[n]{\frac{1-|w|^2/|z|^2+z^{-2n}\overline{w}/\overline{z}}{\sqrt{|z|^{4n}(1-|w|^2/|z|^2)^2-1}}}.
\end{aligned}
\tag{5.61}
$$

Let $\Pi^{(n)}$ denote the corresponding factorization map and $M_\eta^{'(n)} := \Pi^{(n)}\left(M_\eta^{(2n)}\right)$. Then $D_{s,t}^{(n)} := \Pi^{(n)}\left(D_{s,t}^{(4n)}\right)$ is an n-sheeted cover of $D_{s,t}$.

The group $G\left(D_{s,t}^{(2)}\right)$ consists of all maps of the form (5.31) with $A\in SO_{2,1}(\mathbb{R})^0$. We denote this group by $\mathcal{R}^{(1)}$ (observe that $\mathcal{R}^{(1)}$ is isomorphic to $\mathcal{R}_\eta$ – see (5.45)). The $G\left(D_{s,t}^{(2)}\right)$-orbits are the following pairwise CR non-equivalent strongly pseudoconvex hypersurfaces:

$$
\eta_\alpha^{(2)} := \Big\{(z_1,z_2,z_3)\in\mathcal{A}_\eta : |z_1|^2+|z_2|^2-|z_3|^2=\sqrt{(\alpha+1)/2},\Big\},\quad s<\alpha<t
\tag{5.62}
$$

(note that $\eta_\alpha^{(2)}$ is a 2-sheeted cover of η_α). For $n\ge 2$ the group $G\left(D_{s,t}^{(2n)}\right)$ coincides with $\mathcal{R}^{(n)}$ (see (5.56)), where we think of elements of $\mathcal{R}^{(n)}$ as maps defined on $D^{(2n)}_{s,t}$ rather than on $\Omega^{(n)}_{s,t}$. The $G\left(D^{(2n)}_{s,t}\right)$-orbits are the following pairwise CR non-equivalent strongly pseudoconvex hypersurfaces:

$$\eta^{(2n)}_{\alpha} := \left\{(z,w) \in M^{(n)}_{\eta} : |z|^n - |z|^{n-2}|w|^2 = \sqrt{(\alpha+1)/2}\right\}, \quad s < \alpha < t, \tag{5.63}$$

(note that $\eta^{(2n)}_{\alpha}$ is a $2n$-sheeted cover of η_{α}).

Next, the group $G\left(D^{(n)}_{s,t}\right)$ for odd $n \geq 3$ consists of all lifts from the domain $D_{1,\infty}$ to $D^{(n)}_{1,\infty} = M^{'(n)}_{\eta}$ of all elements of $\mathcal{R}_{\eta}$ (see (5.45)). This group is isomorphic to $\mathcal{R}^{(n)}$. Note, however, that the isotropy subgroup of every point under the action of this group on $D^{(n)}_{s,t}$ consists of two points, whereas the action of $\mathcal{R}^{(n)}$ on $D^{(2n)}_{s,t}$ is free (observe also that the isotropy subgroup of every point under the action of $\mathcal{R}_{\eta}$ consists of two points and that the action of $\mathcal{R}^{(1)}$ on $D^{(2)}_{s,t}$ is free). This difference will be important in the proof of Theorem 5.2 (see step (II) of the orbit gluing procedure there).

The $G\left(D^{(n)}_{s,t}\right)$-orbits are the following pairwise CR non-equivalent strongly pseudoconvex hypersurfaces:

$$\eta^{(n)}_{\alpha} := \Pi_n\left(\eta^{(4n)}_{\alpha}\right), \quad s < \alpha < t \tag{5.64}$$

(note that $\eta^{(n)}_{\alpha}$ is an n-sheeted cover of η_{α}).

Finally, the group $G\left(D^{(\infty)}_{s,t}\right)$ coincides with $\mathcal{R}^{(\infty)}$ (see (5.58)), where we think of the elements of $\mathcal{R}^{(\infty)}$ as maps defined on $D^{(\infty)}_{s,t}$ rather than on $\Omega^{(\infty)}_{s,t}$. The $G\left(D^{(\infty)}_{s,t}\right)$-orbits are the following pairwise CR non-equivalent hypersurfaces:

$$\eta^{(\infty)}_{\alpha} := \left\{(z,w) \in M^{(\infty)}_{\eta} : \exp\left(2\mathrm{Re}\, z\right)\left(1-|w|^2\right) = \sqrt{(\alpha+1)/2}\right\}, \quad s < \alpha < t \tag{5.65}$$

(here $\eta^{(\infty)}_{\alpha}$ is an infinitely-sheeted cover of η_{α}).

(11) Here we show how a Levi-flat and complex curve orbit can be attached to some of the domains introduced in **(8)** and **(10)**.

(a) It is straightforward to show from the explicit from of $\Phi^{(n)}$, for $n \geq 2$ (see (5.50)), that the complex structure of $M^{(n)}_{\eta}$ extends to a complex structure on

$$\tilde{\mathcal{A}}^{(n)}_{\eta} := \left\{(\zeta : z : w) \in \mathbb{CP}^2 : |z|^n - |z|^{n-2}|w|^2 > |\zeta|^n\right\}.$$

The set at infinity in $\tilde{\mathcal{A}}^{(n)}_{\eta}$ is

$$\mathcal{O}^{(2n)} := \left\{(0 : z : w) \in \mathbb{CP}^2 : |w| < |z|\right\}, \tag{5.66}$$

and we have $\tilde{\mathcal{A}}_\eta^{(n)} = \mathcal{A}_\eta^{(n)} \cup \mathcal{O}^{(2n)}$ (see (5.51)). Let $\tilde{M}_\eta^{(n)}$ denote $\tilde{\mathcal{A}}_\eta^{(n)}$ with the extended complex structure. In the complex structure of $\tilde{M}_\eta^{(n)}$ the set $\mathcal{O}^{(2n)}$ is a complex curve whose complex structure is identical to that induced from $\mathbb{CP}^2$. The action of the group $\mathcal{R}^{(n)}$ (see (5.56)) extends to an action by holomorphic transformations on $\tilde{M}_\eta^{(n)}$, and $\mathcal{O}^{(2n)}$ is an orbit of this action. The map $\Phi^{(n)}$ has ramification locus on $\mathcal{O}^{(2n)}$ and maps it in a 1-to-1 fashion onto the complex curve

$$\mathcal{O}^{(2)} := \Big\{(0 : z_1 : z_2 : z_3) \in \mathbb{CP}^3 : z_1^2 + z_2^2 - z_3^2 = 0, \\ \operatorname{Im}(z_2(\overline{z_1} + \overline{z_3})) > 0\Big\}. \tag{5.67}$$

Note that $\mathcal{O}^{(2)}$ is an $\mathcal{R}^{(1)}$-orbit (clearly, $\mathcal{R}^{(1)}$ acts on all of $\mathcal{Q}_-$ – see **(10)**).

For $1 \le s < \infty$ and all $n \ge 1$ define

$$D_s^{(2n)} := D_{s,\infty}^{(2n)} \cup \mathcal{O}^{(2n)}. \tag{5.68}$$

The group $G\Big(D_s^{(2n)}\Big)$ (with the exception of the case $n = 1$, $s = 1$) coincides with $\mathcal{R}^{(n)}$ for all n; its orbits in $D_s^{(2n)}$ are the strongly pseudoconvex hypersurfaces $\eta_\alpha^{(2n)}$ for $\alpha > s$ (see (5.63)) and the complex curve $\mathcal{O}^{(2n)}$. The map Ψ_η is a branched covering map on $D_s^{(2)}$, has ramification locus on $\mathcal{O}^{(2)}$, maps it in a 1-to-1 fashion onto the complex curve $\mathcal{O} \subset \mathbb{C}^2$ (see (5.44)), and takes $D_s^{(2)}$ onto D_s. Similarly, for $n \ge 2$, the map $\Psi_\eta \circ \Phi_\eta^{(n)}$ is a branched covering map on $D_s^{(2n)}$, has ramification locus on $\mathcal{O}^{(2n)}$ and takes $D_s^{(2n)}$ onto D_s.

We note that $D_1^{(2)} = \mathcal{A}_\eta \cup \mathcal{O}^{(2)}$ (see (5.53)) is holomorphically equivalent to Δ^2 (see **(11)(c)** below), hence it will be excluded from our considerations.

(b) Fix an odd $n \in \mathbb{N}$, $n \ge 3$, and let $\Gamma^{(n)}$ be the cyclic group of four elements generated by the obvious extension of automorphism (5.61) to $\tilde{M}_\eta^{(2n)} = D_1^{(4n)}$. The group $\Gamma^{(n)}$ acts freely properly discontinuously on $M_\eta^{(2n)} \subset \tilde{M}_\eta^{(2n)}$ and fixes every point in $\mathcal{O}^{(4n)}$. It is straightforward to show that the orbifold obtained by taking the quotient of $\tilde{M}_\eta^{(2n)}$ by the action of $\Gamma^{(n)}$ can in fact be given the structure of a complex manifold (we denote it by $\tilde{M}_\eta^{'(n)}$) that extends the structure of $M_\eta^{'(n)}$ (see **(10)**). The extension of the map $\Pi^{(n)}$ (see **(10)**) is holomorphic on all of $\tilde{M}_\eta^{(2n)}$, has ramification locus on $\mathcal{O}^{(4n)}$ and maps $\mathcal{O}^{(4n)}$ onto a complex curve $\mathcal{O}^{(n)} \subset \tilde{M}_\eta^{'(n)}$ in a 1-to-1 fashion (note that $\tilde{M}_\eta^{'(n)} = M_\eta^{'(n)} \cup \mathcal{O}^{(n)}$). The covering map from $M_\eta^{'(n)}$ onto $D_{1,\infty}$ (see (5.43)) extends to a branched covering map from $\tilde{M}_\eta^{'(n)}$ onto D_1 (see (5.47)) with ramification locus $\mathcal{O}^{(n)}$, and takes $\mathcal{O}^{(n)}$ onto $\mathcal{O}$ (see (5.44)) in a 1-to-1 fashion.

For $1 \le s < \infty$ define

$$D_s^{(n)} := \Pi^{(n)}\left(D_s^{(4n)}\right). \tag{5.69}$$

The group $G\left(D_s^{(n)}\right)$ is isomorphic to $\mathcal{R}^{(n)}$ and consists of the extensions from $D_{s,\infty}^{(n)} = D_s^{(n)} \setminus \mathcal{O}^{(n)}$ to $D_s^{(n)}$ of all elements of the group $G\left(D_{s,\infty}^{(n)}\right)$. The $G\left(D_s^{(n)}\right)$-orbits are the strongly pseudoconvex hypersurfaces $\eta_\alpha^{(n)}$ with $\alpha > s$ (see (5.64)) and the complex curve $\mathcal{O}^{(n)}$.

(c) Define

$$M^{(1)} := \Psi_\nu^{-1}(\Omega_1) \cup D_1^{(2)} \cup \mathcal{O}_0^{(1)}, \tag{5.70}$$

where

$$\begin{aligned}\mathcal{O}_0^{(1)} := \Big\{(z_1, z_2, z_3) \in \mathbb{C}^3 \setminus \mathbb{R}^3 : |iz_1 + z_2| = |iz_3 - 1|, \\ |iz_1 - z_2| = |iz_3 + 1|, \operatorname{Im} z_3 < 0\Big\} \cap \mathcal{Q}_-\end{aligned} \tag{5.71}$$

(see **(10)** for the definition of Ψ_ν and (5.48) for the definition of $\mathcal{Q}_-$). Clearly, $M^{(1)}$ is invariant under the action of the group $\mathcal{R}^{(1)}$ (defined in **(10)**) on $\mathcal{Q}_-$. We will now describe the orbits of the $\mathcal{R}^{(1)}$-action on $M^{(1)}$. The hypersurfaces $\eta_\alpha^{(2)}$ for $\alpha > 1$ (see (5.62)) and

$$\nu_\alpha^{(1)} := \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 - |z_3|^2 = \alpha\right\} \cap \mathcal{Q}_-$$

for $-1 < \alpha < 1$ are strongly pseudoconvex orbits (note that $\nu_\alpha^{(1)}$ is equivalent to ν_α (see (5.40)) by means of the map Ψ_ν); the hypersurface $\mathcal{O}_0^{(1)}$ is the unique Levi-flat orbit; the surfaces

$$\mathcal{O}_6 := \Big\{(z_1, z_2, z_3) \in i\mathbb{R}^3 : \operatorname{Im} z_3 < 0\Big\} \cap \mathcal{Q}_- \tag{5.72}$$

and $\mathcal{O}^{(2)}$ are codimension 2 totally real and complex curve orbits, respectively (observe that $\Psi_\nu^{-1}(\Omega_1) = \mathcal{A}_\nu \cup \mathcal{O}_6$ (see (5.52)) with $\mathcal{O}_6 = \Psi_\nu^{-1}(\mathcal{O}_5)$ (see (5.42))).

The manifold $M^{(1)}$ can be mapped onto $\Delta \times \mathbb{CP}^1 \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ by the inverse to a variant of the Segre map.[1] Let $[(Z_0 : Z_1), (W_0 : W_1)]$ denote two pairs of homogeneous coordinates in $\mathbb{CP}^1 \times \mathbb{CP}^1$, where the infinity in $\mathbb{CP}^1$ is given by the vanishing of the coordinate that carries index 0. Consider the following map $\mathcal{S}$ from $\mathbb{CP}^1 \times \mathbb{CP}^1$ to $\mathbb{CP}^3$:

$$\begin{aligned} z_0 &= i\left(Z_0W_0 - Z_1W_1\right), \\ z_1 &= Z_0W_1 + Z_1W_0, \\ z_2 &= i\left(Z_0W_1 - Z_1W_0\right), \\ z_3 &= Z_0W_0 + Z_1W_1. \end{aligned}$$

[1] We are grateful to Stefan Nemirovski for showing us this realization of $M^{(1)}$.

It is straightforward to see that this map takes $\Delta\times\mathbb{CP}^1$ biholomorphically onto $M^{(1)}$. Under the inverse map $\mathcal{S}^{-1}$ the action of $\mathcal{R}^{(1)}$ on $M^{(1)}$ is transformed into the following action of $SU_{1,1}/\{\pm\mathrm{id}\}\simeq\mathcal{R}^{(1)}$ on $\Delta\times\mathbb{CP}^1$: the element $g\{\pm\mathrm{id}\}\in SU_{1,1}/\{\pm\mathrm{id}\}$ acts on the vector $(Z_0:Z_1)$ by applying the matrix g to the vector and on the vector $(W_0:W_1)$ by applying the matrix $\overline{g}$ to it. The map $\mathcal{S}^{-1}$ takes the orbit $\mathcal{O}_0^{(1)}$ into the $SU_{1,1}/\{\pm\mathrm{id}\}$-orbit $\Delta\times\partial\Delta$, the orbit $\mathcal{O}_6$ into

$$\left\{\left[(1:Z)\,,\left(1:\overline{Z}\right)\right],\,|Z|<1\right\},$$

and the orbit $\mathcal{O}^{(2)}$ into

$$\{[(1:Z)\,,(1:1/Z)]\,,\,0<|Z|<1\}\cup\{[(1:0)\,,(0:1)]\}\,.$$

The domain $\Psi_\nu^{-1}(\Omega_1)$ is mapped by $\mathcal{S}^{-1}$ onto $\Delta\times\Delta$ and $D_1^{(2)}$ onto

$$\Delta\times\Big(\{(1:W),\,|W|>1\}\cup\{(0:1)\}\Big)$$

(hence each of Ω_1, $D_1^{(2)}$ is equivalent to Δ^2). For more general examples of this kind arising from actions of non-compact forms of complex reductive groups see [AG], [FH].

It is clear from the above description of $M^{(1)}$ that in order to obtain a hyperbolic $\mathcal{R}^{(1)}$-invariant submanifold of $M^{(1)}$ containing the Levi-flat orbit $\mathcal{O}_0^{(1)}$, one must remove from $M^{(1)}$ an $\mathcal{R}^{(1)}$-invariant neighborhood of either $\mathcal{O}_6$ or $\mathcal{O}^{(2)}$. Namely, each of the domains

$$\begin{aligned}
\mathfrak{D}_s^{(1)} &:= \Psi_\nu^{-1}(\Omega_{s,1})\cup D_1^{(2)}\cup\mathcal{O}_0^{(1)}, \quad -1<s<1,\\
\hat{\mathfrak{D}}_t^{(1)} &:= \Psi_\nu^{-1}(\Omega_1)\cup D_{1,t}^{(2)}\cup\mathcal{O}_0^{(1)}, \quad 1<t<\infty,\\
\mathfrak{D}_{s,t}^{(1)} &:= \Psi_\nu^{-1}(\Omega_{s,1})\cup D_{1,t}^{(2)}\cup\mathcal{O}_0^{(1)}, \quad -1\le s<1<t\le\infty,\\
&\qquad \text{where } s=-1 \text{ and } t=\infty \text{ do not hold simultaneously,}
\end{aligned}\tag{5.73}$$

is a (2,3)-manifold of this kind (see (5.37), (5.41), (5.60)). Observe here that

$$\Psi_\nu^{-1}(\Omega_{s,1})=\Big\{(z_1,z_2,z_3)\in\mathbb{C}^3: s<|z_1|^2+|z_2|^2-|z_3|^2<1,\\ \mathrm{Im}\,z_3<0\Big\}\cap\mathcal{Q}_-.$$

Each of the groups $G\left(\mathfrak{D}_s^{(1)}\right)=\mathrm{Aut}\left(\mathfrak{D}_s^{(1)}\right)$, $G\left(\hat{\mathfrak{D}}_t^{(1)}\right)=\mathrm{Aut}\left(\hat{\mathfrak{D}}_t^{(1)}\right)$, $G\left(\mathfrak{D}_{s,t}^{(1)}\right)=\mathrm{Aut}\left(\mathfrak{D}_{s,t}^{(1)}\right)$ coincides with $\mathcal{R}^{(1)}$.

(d) We now consider covers of

$$\mathfrak{D}_{-1,\infty}^{(1)}:=\Psi_\nu^{-1}(\Omega_{-1,1})\cup D_{1,\infty}^{(2)}\cup\mathcal{O}_0^{(1)}.\tag{5.74}$$

For $n\ge2$ the domain $\Sigma\backslash\mathcal{O}^{(2n)}$ (see (5.49)) is an n-sheeted cover of $\mathfrak{D}_{-1,\infty}^{(1)}$ with covering map $\Phi^{(n)}$. We equip $\Sigma\setminus\mathcal{O}^{(2n)}$ with the pull-back complex structure under $\Phi^{(n)}$. This complex structure extends the structure of each of $M_\nu^{(n)}$, $M_\eta^{(n)}$ (see **(10)**) and can be extended to a complex structure on all of Σ. Let $M^{(n)}$ be the domain Σ with this extended complex structure. The map $\Phi^{(n)}$ takes $M^{(n)}$ onto

$$\Psi_\nu^{-1}(\Omega_{-1,1}) \cup D_1^{(2)} \cup \mathcal{O}_0^{(1)},$$

has ramification locus on $\mathcal{O}^{(2n)}$ and maps it in a 1-to-1 fashion onto $\mathcal{O}^{(2)}$.

Clearly, the group $\mathcal{R}^{(n)}$ acts on $M^{(n)}$. We will now describe the orbits of this action. The hypersurfaces $\nu_\alpha^{(n)}$ for $-1 < \alpha < 1$ (see (5.57)) and $\eta_\alpha^{(2n)}$ for $\alpha > 1$ (see (5.63)) are strongly pseudoconvex orbits; the hypersurface

$$\mathcal{O}_0^{(n)} := \left\{(z,w) \in M^{(n)} : |z|^n - |z|^{n-2}|w|^2 = 1\right\} \tag{5.75}$$

is the unique Levi-flat orbit (CR-equivalent to $\Delta \times \partial\Delta$ for every n and covering $\mathcal{O}_0^{(1)}$ by means of the n-to-1 map $\Phi^{(n)}$); the complex curve $\mathcal{O}^{(2n)}$ is the unique codimension 2 orbit.

We now introduce the following domains in $M^{(n)}$:

$$\begin{aligned}
\mathfrak{D}_s^{(n)} &:= \left\{(\zeta : z : w) \in M^{(n)} : |z|^n - |z|^{n-2}|w|^2 > \sqrt{(s+1)/2}|\zeta|^n\right\} = \\
&\quad \Omega_{s,1}^{(n)} \cup D_1^{(2n)} \cup \mathcal{O}_0^{(n)},\ -1 < s < 1,\\
\mathfrak{D}_{s,t}^{(n)} &:= \left\{(\zeta : z : w) \in M^{(n)} : \sqrt{(s+1)/2}|\zeta|^n < |z|^n - |z|^{n-2}|w|^2 < \right.\\
&\quad \left.\sqrt{(t+1)/2}|\zeta|^n\right\} = \Omega_{s,1}^{(n)} \cup D_{1,t}^{(2n)} \cup \mathcal{O}_0^{(n)},\ -1 \le s < 1 < t \le \infty,\\
&\quad \text{where } s = -1 \text{ and } t = \infty \text{ do not hold simultaneously}
\end{aligned}$$

(see (5.55), (5.60)). Each of these domains is a (2,3)-manifold. Each of the groups $G\left(\mathfrak{D}_s^{(n)}\right) = \mathrm{Aut}\left(\mathfrak{D}_s^{(n)}\right)$, $G\left(\mathfrak{D}_{s,t}^{(n)}\right) = \mathrm{Aut}\left(\mathfrak{D}_{s,t}^{(n)}\right)$ coincides with $\mathcal{R}^{(n)}$.

Next, $\mathbb{C} \times \Delta$ is an infinitely-sheeted cover of $\mathfrak{D}_{-1,\infty}^{(1)}$ with covering map Λ (see (5.54)). We equip the domain $\mathbb{C} \times \Delta$ with the pull-back complex structure under Λ and denote the resulting manifold by $M^{(\infty)}$. Clearly, the complex structure of $M^{(\infty)}$ extends the structure of each of $M_\nu^{(\infty)}$, $M_\eta^{(\infty)}$ (see **(10)**). The group $\mathcal{R}^{(\infty)}$ (see (5.58)) acts on $M^{(\infty)}$, and the orbits of this action are the strongly pseudoconvex hypersurfaces $\nu_\alpha^{(\infty)}$ for $-1 < \alpha < 1$ (see (5.59)) and $\eta_\alpha^{(\infty)}$ for $\alpha > 1$ (see (5.65)), as well as the Levi-flat hypersurface

$$\mathcal{O}_0^{(\infty)} := \left\{(z,w) \in M^{(\infty)} : \exp(2\mathrm{Re}\, z)(1 - |w|^2) = 1\right\} \tag{5.76}$$

(note that $\mathcal{O}_0^{(\infty)}$ is CR-equivalent to $\mathcal{O}_1$ – see (5.4)).

We now introduce the following domains in $M^{(\infty)}$:

$$\mathfrak{D}_s^{(\infty)} := \left\{ (z,w) \in M^{(\infty)} : \exp\left(2\mathrm{Re}\, z\right)\left(1-|w|^2\right) > \sqrt{(s+1)/2} \right\} =$$
$$\Omega_{s,1}^{(\infty)} \cup D_{1,\infty}^{(\infty)} \cup \mathcal{O}_0^{(\infty)},\ -1 < s < 1,$$
$$\mathfrak{D}_{s,t}^{(\infty)} := \left\{ (z,w) \in M^{(\infty)} : \sqrt{(s+1)/2} < \exp\left(2\mathrm{Re}\, z\right)\left(1-|w|^2\right) < \right.$$
$$\left. \sqrt{(t+1)/2} \right\} = \Omega_{s,1}^{(\infty)} \cup D_{1,t}^{(\infty)} \cup \mathcal{O}_0^{(\infty)},\ -1 \le s < 1 < t \le \infty,$$
where $s = -1$ and $t = \infty$ do not hold simultaneously

(see (5.55), (5.60)). Each of these domains is a (2,3)-manifold. The groups $G\left(\mathfrak{D}_s^{(\infty)}\right) = \mathrm{Aut}\left(\mathfrak{D}_s^{(\infty)}\right)$, $G\left(\mathfrak{D}_{s,t}^{(\infty)}\right) = \mathrm{Aut}\left(\mathfrak{D}_{s,t}^{(\infty)}\right)$ all coincide with $\mathcal{R}^{(\infty)}$.

5.2 Strongly Pseudoconvex Orbits

In this section we give a complete classification of (2,3)-manifolds M for which every $G(M)$-orbit is a strongly pseudoconvex real hypersurface in M. In the formulation below we use the notation introduced in the previous section.

Theorem 5.2. ([I5])*Let M be a (2,3)-manifold. Assume that the $G(M)$-orbit of every point in M is a strongly pseudoconvex real hypersurface. Then M is holomorphically equivalent to one of the following manifolds:*

(i) $R_{\alpha,s,t}$, $\alpha \in \mathbb{R}$, $|\alpha| \ge 1$, $\alpha \ne 1$, *with either* $s = 0$, $t = 1$, *or* $s = 1$, $1 < t \le \infty$;
(ii) $U_{s,t}$, *with either* $s = 0$, $t = 1$, *or* $s = 1$, $1 < t \le \infty$;
(iii) $\mathfrak{S}_{s,t}$, *with either* $s = 0$, $t = 1$, *or* $s = 1$, $1 < t < \infty$;
(iv) $\mathfrak{S}_{s,t}^{(\infty)}$, *with either* $s = 0$, $t = 1$, *or* $s = 1$, $1 < t < \infty$;
(v) $\mathfrak{S}_{s,t}^{(n)}$, $n \ge 2$, *with either* $s = 0$, $t = 1$, *or* $s = 1$, $1 < t < \infty$;
(vi) $V_{\alpha,s,1}$, $\alpha > 0$, *with* $e^{-2\pi\alpha} < s < 1$;
(vii) $E_{s,t}$, *with* $1 \le s < t < \infty$;
(viii) $E_{s,t}^{(4)}$, *with* $1 \le s < t < \infty$;
(ix) $E_{s,t}^{(2)}$, *with* $1 \le s < t < \infty$;
(x) $\Omega_{s,t}$, *with* $-1 \le s < t \le 1$;
(xi) $\Omega_{s,t}^{(\infty)}$, *with* $-1 \le s < t \le 1$;
(xii) $\Omega_{s,t}^{(n)}$, $n \ge 2$, *with* $-1 \le s < t \le 1$;
(xiii) $D_{s,t}$, *with* $1 \le s < t \le \infty$;
(xiv) $D_{s,t}^{(\infty)}$, *with* $1 \le s < t \le \infty$;
(xv) $D_{s,t}^{(n)}$, $n \ge 2$, *with* $1 \le s < t \le \infty$.

The manifolds on list (i)–(xv) are pairwise holomorphically non-equivalent.

Proof: In [CaE1] E. Cartan classified all homogeneous 3-dimensional strongly pseudoconvex CR-manifolds. Since the $G(M)$-orbit of every point in M is such a manifold, every $G(M)$-orbit is CR-equivalent to a manifold on Cartan's list. We reproduce Cartan's classification below together with the corresponding groups of CR-automorphisms. Note that all possible covers of the hypersurfaces χ, μ_α, ν_α and η_α appear below as explicitly realized in [I4].

(a) S^3;
(b) $\mathcal{L}_m = S^3/\mathbb{Z}_m$, $m \in \mathbb{N}, m \geq 2$;
(c) $\sigma = \left\{(z,w) \in \mathbb{C}^2 : \operatorname{Re} w = |z|^2\right\}$ (cf. (3.3));
(d) $\varepsilon'_\alpha := \left\{(z,w) \in \mathbb{C}^2 : \operatorname{Re} z = |w|^\alpha,\ w \neq 0\right\}$, $\alpha > 0$ (cf. (3.3));
(e) $\omega' := \left\{(z,w) \in \mathbb{C}^2 : \operatorname{Re} z = \exp\left(\operatorname{Re} w\right)\right\}$ (cf. (3.3));
(f) $\delta = \left\{(z,w) \in \mathbb{C}^2 : |w| = \exp\left(|z|^2\right)\right\}$ (cf. (3.3));
(g) τ_α, $\alpha \in \mathbb{R}$, $|\alpha| \geq 1$, $\alpha \neq 1$ (see (5.3));
(h) ξ (see (5.8));
(j) χ (see (5.11));
(j') $\chi^{(\infty)}$ (see (5.19));
(j") $\chi^{(n)}$, $n \geq 2$ (see (5.17));
(k) ρ_α, $\alpha > 0$ (see (5.22));
(l) μ_α, $\alpha > 1$ (see (5.25));
(l') $\mu_\alpha^{(4)}$, $\alpha > 1$ (see (5.34));
(l") $\mu_\alpha^{(2)}$, $\alpha > 1$ (see (5.32));
(m) ν_α, $-1 < \alpha < 1$, (see (5.40));
(m') $\nu_\alpha^{(\infty)}$, $-1 < \alpha < 1$, (see (5.59));
(m") $\nu_\alpha^{(n)}$, $-1 < \alpha < 1$, $n \geq 2$ (see (5.57));
(n) η_α, $\alpha > 1$, (see (5.46));
(n') $\eta_\alpha^{(\infty)}$, $\alpha > 1$, (see (5.65));
(n") $\eta_\alpha^{(n)}$, $\alpha > 1$, $n \geq 2$ (see (5.62), (5.63), (5.64)).

The above hypersurfaces are pairwise CR non-equivalent. The corresponding groups of CR-automorphisms are as follows:

<u>(a) $\operatorname{Aut}_{CR}(S^3)$</u> : maps of the form (5.38), where the matrix Q defined in (5.39) belongs to $SU_{2,1}$;

<u>(b) $\operatorname{Aut}_{CR}(\mathcal{L}_m)$, $m \geq 2$</u> :

$$\left[\begin{pmatrix} z \\ w \end{pmatrix}\right] \mapsto \left[U \begin{pmatrix} z \\ w \end{pmatrix}\right],$$

where $U \in U_2$, and $[(z,w)] \in \mathcal{L}_m$ denotes the equivalence class of $(z,w) \in S^3$ under the action of $\mathbb{Z}_m$ embedded in U_2 as a subgroup of scalar matrices;

(c) $\mathrm{Aut}_{CR}(\sigma)$:

$$\begin{aligned} z &\mapsto \lambda e^{i\psi} z + a, \\ w &\mapsto \lambda^2 w + 2\lambda e^{i\psi}\overline{a} z + |a|^2 + i\gamma, \end{aligned} \tag{5.77}$$

where $\lambda > 0$, $\psi, \gamma \in \mathbb{R}$, $a \in \mathbb{C}$ (cf. (3.7));

(d) $\mathrm{Aut}_{CR}(\varepsilon'_\alpha)$:

$$\begin{aligned} z &\mapsto \frac{\lambda z + i\beta}{i\mu z + \kappa}, \\ w &\mapsto \frac{e^{i\psi}}{(i\mu z + \kappa)^{2/\alpha}} w, \end{aligned} \tag{5.78}$$

where $\lambda, \beta, \mu, \kappa, \psi \in \mathbb{R}$, $\lambda\kappa + \mu\beta = 1$ (cf. (3.1));

(e) $\mathrm{Aut}_{CR}(\omega')$:

$$\begin{aligned} z &\mapsto \frac{\lambda z + i\beta}{i\mu z + \kappa}, \\ w &\mapsto w - 2\ln(i\mu z + \kappa) + i\gamma, \end{aligned} \tag{5.79}$$

where $\lambda, \beta, \mu, \kappa, \gamma \in \mathbb{R}$, $\lambda\kappa + \mu\beta = 1$ (cf. (3.11));

(f) $\mathrm{Aut}_{CR}(\delta)$:

$$\begin{aligned} z &\mapsto e^{i\psi} z + a, \\ w &\mapsto e^{i\theta} \exp\Big(2e^{i\psi}\overline{a} z + |a|^2\Big) w, \end{aligned} \tag{5.80}$$

where $\psi, \theta \in \mathbb{R}$, $a \in \mathbb{C}$ (cf. (3.13));

(g) $\mathrm{Aut}_{CR}(\tau_\alpha)$: the group G_α (see (5.2));

(h) $\mathrm{Aut}_{CR}(\xi)$: the group $\mathfrak{G}$ (see (5.7));

(j) $\mathrm{Aut}_{CR}(\chi)$ is generated by $\mathcal{R}_\chi$ (see (5.10)) and the map

$$\begin{aligned} z &\mapsto z, \\ w &\mapsto -w; \end{aligned} \tag{5.81}$$

(j') $\mathrm{Aut}_{CR}\left(\chi^{(\infty)}\right)$ is generated by maps (5.18) and the map

$$\begin{aligned} z &\mapsto \overline{z}, \\ w &\mapsto \overline{w}; \end{aligned}$$

(j”) $\mathrm{Aut}_{CR}\left(\chi^{(n)}\right)$, $n \geq 2$, is generated by maps (5.16) and map (5.81);

(k) $\mathrm{Aut}_{CR}(\rho_\alpha)$: see (5.21);

(l) $\mathrm{Aut}_{CR}(\mu_\alpha)$: the group $\mathcal{R}_\mu$ (see (5.24));

(l’) $\mathrm{Aut}_{CR}\left(\mu_\alpha^{(4)}\right)$ is generated by maps (5.33) and the map

$$
\begin{aligned}
z &\mapsto i\frac{z(|z|^2+|w|^2)-\overline{w}}{\sqrt{1+(|z|^2+|w|^2)^2}},\\
w &\mapsto i\frac{w(|z|^2+|w|^2)+\overline{z}}{\sqrt{1+(|z|^2+|w|^2)^2}};
\end{aligned}
$$

(l”) $\mathrm{Aut}_{CR}\left(\mu_\alpha^{(2)}\right)$ is generated by $\mathcal{R}_\mu^{(2)}$ (see (5.31)) and the map

$$
\begin{aligned}
z_1 &\mapsto -z_1,\\
z_2 &\mapsto -z_2,\\
z_3 &\mapsto -z_3;
\end{aligned}
\tag{5.82}
$$

(m) $\mathrm{Aut}_{CR}(\nu_\alpha)$ is generated by $\mathcal{R}_\nu$ (see (5.38)) and map (5.81);

(m’) $\mathrm{Aut}_{CR}\left(\nu_\alpha^{(\infty)}\right)$ is generated by $\mathcal{R}^{(\infty)}$ (see (5.58)) and the map

$$
\begin{aligned}
z &\mapsto \overline{z}+\ln\left(-\frac{1+e^{2z}w(1-|w|^2)}{\sqrt{1-\exp\left(4\mathrm{Re}\,z\right)\left(1-|w|^2\right)^2}}\right),\\
w &\mapsto -\frac{\overline{w}+e^{2z}(1-|w|^2)}{1+e^{2z}w(1-|w|^2)};
\end{aligned}
$$

(m”) $\mathrm{Aut}_{CR}\left(\nu_\alpha^{(n)}\right)$ $n \geq 2$ is generated by $\mathcal{R}^{(n)}$ (see (5.56)) and the map

$$z \mapsto \overline{z}\sqrt[n]{\frac{\left(1+z^{n-1}w(1-|w|^2/|z|^2)\right)^2}{1-|z|^{2n}(1-|w|^2/|z|^2)^2}},$$

$$w \mapsto -\frac{\overline{w}/\overline{z}+z^n(1-|w|^2/|z|^2)}{1+z^{n-1}w(1-|w|^2/|z|^2)}\times$$

$$\overline{z}\sqrt[n]{\frac{\left(1+z^{n-1}w(1-|w|^2/|z|^2)\right)^2}{1-|z|^{2n}(1-|w|^2/|z|^2)^2}};$$

<u>(n) $\mathrm{Aut}_{CR}(\eta_\alpha)$</u> : the group $\mathcal{R}_\eta$ (see (5.45));

<u>(n') $\mathrm{Aut}_{CR}\left(\eta_\alpha^{(\infty)}\right)$</u> is generated by $\mathcal{R}^{(\infty)}$ (see (5.58)) and the map

$$z \mapsto 2z+\overline{z}+\ln\left(i\frac{1-|w|^2+e^{-2z}\overline{w}}{\sqrt{\exp\left(4\mathrm{Re}\, z\right)(1-|w|^2)^2-1}}\right),$$

$$w \mapsto \frac{1+e^{2z}w(1-|w|^2)}{\overline{w}+e^{2z}(1-|w|^2)};$$

<u>(n") $\mathrm{Aut}_{CR}\left(\eta_\alpha^{(2)}\right)$</u> is generated by $\mathcal{R}^{(1)}$ (see **(10)**) and map (5.82);

<u>(n") $\mathrm{Aut}_{CR}\left(\eta_\alpha^{(2n)}\right)$ $n \geq 2$</u> is generated by $\mathcal{R}^{(n)}$ (see (5.56)) and the map

$$z \mapsto z^2\overline{z}\sqrt[n]{\frac{\left(1-|w|^2/|z|^2+z^{-n}\overline{w}/\overline{z}\right)^2}{|z|^{2n}(1-|w|^2/|z|^2)^2-1}},$$

$$w \mapsto \frac{1+z^{n-1}w(1-|w|^2/|z|^2)}{\overline{w}/\overline{z}+z^n(1-|w|^2/|z|^2)}\times$$

$$z^2\overline{z}\sqrt[n]{\frac{\left(1-|w|^2/|z|^2+z^{-n}\overline{w}/\overline{z}\right)^2}{|z|^{2n}(1-|w|^2/|z|^2)^2-1}};$$

<u>(n") $\mathrm{Aut}_{CR}\left(\eta_\alpha^{(n)}\right)$, $n \geq 3$ is odd</u> : this group is isomorphic to $\mathcal{R}^{(n)}$ and consists of all lifts from the domain $D_{1,\infty}$ (see (5.43)) to $M_\eta^{'(n)}$ (see **(10)**) of maps from $\mathcal{R}_\eta$ (see (5.45)).

Fix $p \in M$ and suppose that $O(p)$ is CR-equivalent to $\mathfrak{m}$, where $\mathfrak{m}$ is one of the hypersurfaces listed above in (a)–(n"). As in Chaps. 3, 4, we say in this case that $\mathfrak{m}$ is the model for $O(p)$. Since $G(M)$ acts properly and effectively on $O(p)$, the CR-equivalence induces an isomorphism between $G(M)$ and a closed connected 3-dimensional subgroup $R_{\mathfrak{m}}$ of the Lie group $\mathrm{Aut}_{CR}(\mathfrak{m})$ that acts properly and transitively on $\mathfrak{m}$. We will now prove an analogue of Propositions 3.4 and 4.5. In what follows $\mathcal{P}$ denotes the right half-plane $\{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$.

Proposition 5.3. ([I5]) *We have*

(A) $R_{\mathfrak{m}} = Aut_{CR}(\mathfrak{m})^0$, *if* $\mathfrak{m}$ *is one of (g)-(n");*
(B) R_{S^3} *is conjugate in* $Aut_{CR}(S^3)$ *to* SU_2;
(C) $R_{\mathcal{L}_m} = SU_2/(SU_2 \cap \mathbb{Z}_m)$, $m \geq 2$;
(D) R_σ *is the Heisenberg group* N*, that is, it consists of all elements of* $\mathrm{Aut}_{CR}(\sigma)$ *with* $\lambda = 1$, $\psi = 0$ *in formula (5.77) (cf. (3.7));*
(E) $R_{\varepsilon'_\alpha}$ *either is the subgroup of* $Aut_{CR}(\varepsilon'_\alpha)$ *corresponding to a subgroup of* $Aut(\mathcal{P})$*, conjugate in* $Aut(\mathcal{P})$ *to the subgroup* $\mathcal{T}$ *given by*

$$z \mapsto \lambda z + i\beta, \tag{5.83}$$

where $\lambda > 0$, $\beta \in \mathbb{R}$*, or, for* $\alpha \in \mathbb{Q}$*, is the subgroup* $\mathfrak{V}_\alpha$ *given by* $\psi = 0$ *in formula (5.78);*
(F) $R_{\omega'}$ *either is the subgroup of* $Aut_{CR}(\omega')$ *corresponding to a subgroup of* $Aut(\mathcal{P})$ *conjugate in* $Aut(\mathcal{P})$ *to the subgroup* $\mathcal{T}$ *specified in (E), or is the subgroup* $\mathfrak{V}_\infty$ *given by* $\gamma = 0$ *in formula (5.79);*
(G) R_δ *coincides with the subgroup of* $Aut_{CR}(\delta)$ *given by* $\psi = 0$ *in formula (5.80).*

Proof: Statement (A) is clear since in (g)-(n") we have $\dim \mathrm{Aut}_{CR}(\mathfrak{m}) = d(M) = 3$. Furthermore, statements (B), (C), (D) and (G) are proved as statements (i), (ii), (iii) of Proposition 4.5.

To prove (E) and (F) we observe that every codimension 1 subgroup of $\mathrm{Aut}(\mathcal{P})$ is conjugate in $\mathrm{Aut}(\mathcal{P})$ to the subgroup $\mathcal{T}$ defined in (5.83). The only codimension 1 subgroups of $\mathrm{Aut}_{CR}(\varepsilon'_\alpha)$ and $\mathrm{Aut}_{CR}(\omega')$ that do not arise from codimension 1 subgroups of $\mathrm{Aut}(\mathcal{P})$ are $\mathfrak{V}_\alpha$ and $\mathfrak{V}_\infty$, respectively. Note that $\mathfrak{V}_\alpha$ is not closed in $\mathrm{Aut}_{CR}(\varepsilon'_\alpha)$ unless $\alpha \in \mathbb{Q}$.

The proposition is proved. ■

Proposition 5.3 implies, in particular, that if for some point $p \in M$ the model for $O(p)$ is S^3, then M admits an effective action of SU_2 by holomorphic transformations, and therefore is holomorphically equivalent to one of the manifolds listed in [IKru2] (see Sect. 6.3). However, none of the (2,3)-manifolds on the list has a spherical orbit. Hence S^3 is not the model for any orbit in M.

We now observe – directly from the explicit forms of the CR-automorphism groups of the models listed above – that for each $\mathfrak{m}$ every element of $\mathrm{Aut}_{CR}(\mathfrak{m})$

extends to a holomorphic automorphism of a certain complex manifold $M_{\mathfrak{m}}$ containing $\mathfrak{m}$, such that every $R_{\mathfrak{m}}$-orbit O in $M_{\mathfrak{m}}$ is strongly pseudoconvex and exactly one of the following holds: (a) O is CR-equivalent to $\mathfrak{m}$ (cases (b)–(k)); (b) O belongs to the same family to which $\mathfrak{m}$ belongs and the $R_{\mathfrak{m}}$-orbits are pairwise CR non-equivalent (cases (l)–(n")). The manifolds $M_{\mathfrak{m}}$ are as follows:

(b) $M_{\mathcal{L}_m} = \mathbb{C}^2 \setminus \{0\}/\mathbb{Z}_m$, $m \geq 2$;

(c) $M_\sigma = \mathbb{C}^2$;

(d) $M_{\varepsilon'_\alpha} = \big\{(z,w) \in \mathbb{C}^2 : \operatorname{Re} z > 0,\ w \neq 0\big\}$;

(e) $M_{\omega'} = \big\{(z,w) \in \mathbb{C}^2 : \operatorname{Re} z > 0\big\}$;

(f) $M_\delta = \big\{(z,w) \in \mathbb{C}^2 : w \neq 0\big\}$;

(g) $M_{\tau_\alpha} = \big\{(z,w) \in \mathbb{C}^2 : \operatorname{Re} z > 0,\ \operatorname{Re} w > 0\big\}$;

(h) $M_\xi = \big\{(z,w) \in \mathbb{C}^2 : \operatorname{Re} w > 0\big\}$;

(j) $M_\chi = \mathbb{C}^2 \setminus \big\{(z,w) \in \mathbb{C}^2 : \operatorname{Re} z = 0,\ \operatorname{Re} w = 0\big\}$;

(j') $M_{\chi^{(\infty)}} = M_\chi^{(\infty)}$ (see **(4)**);

(j") $M_{\chi^{(n)}} = M_\chi^{(n)}$, $n \geq 2$ (see **(4)**);

(k) $M_{\rho_\alpha} = \mathbb{C}^2 \setminus \big\{(z,w) \in \mathbb{C}^2 : \operatorname{Re} z = 0,\ \operatorname{Re} w = 0\big\}$;

(l) $M_{\mu_\alpha} = \bigcup_{\alpha>1} \mu_\alpha = \mathbb{CP}^2 \setminus \mathbb{RP}^2$;

(l') $M_{\mu_\alpha^{(4)}} = \bigcup_{\alpha>1} \mu_\alpha^{(4)} = M_\mu^{(4)}$ (see **(7)**);

(l") $M_{\mu_\alpha^{(2)}} = \bigcup_{\alpha>1} \mu_\alpha^{(2)} = \mathcal{Q}_+ \setminus \mathbb{R}^3$ (see (5.28));

(m) $M_{\nu_\alpha} = \bigcup_{-1<\alpha<1} \nu_\alpha = \Omega_{-1,1}$ (see (5.37));

(m') $M_{\nu_\alpha^{(\infty)}} = \bigcup_{-1<\alpha<1} \nu_\alpha^{(\infty)} = M_\nu^{(\infty)} = \Omega_{-1,1}^{(\infty)}$ (see (5.55));

(m") $M_{\nu_\alpha^{(n)}} = \bigcup_{-1<\alpha<1} \nu_\alpha^{(n)} = M_\nu^{(n)} = \Omega_{-1,1}^{(n)}$, $n \geq 2$ (see (5.55));

(n) $M_{\eta_\alpha} = \bigcup_{\alpha>1} \eta_\alpha = D_{1,\infty}$ (see (5.43));

(n') $M_{\eta_\alpha^{(\infty)}} = \bigcup_{\alpha>1} \eta_\alpha^{(\infty)} = M_\eta^{(\infty)} = D_{1,\infty}^{(\infty)}$ (see (5.60));

(n") $M_{\eta_\alpha^{(2)}} = \bigcup_{\alpha>1} \eta_\alpha^{(2)} = \mathcal{A}_\eta$ (see (5.53));

(n") $M_{\eta_\alpha^{(2n)}} = \bigcup_{\alpha>1} \eta_\alpha^{(2n)} = M_\eta^{(n)} = D_{1,\infty}^{(2n)}$, $n \geq 2$ (see (5.60));

(n") $M_{\eta_\alpha^{(n)}} = \bigcup_{\alpha>1} \eta_\alpha^{(n)} = M_\eta^{'(n)} = D_{1,\infty}^{(n)}$, $n \geq 3$ is odd (see **(10)**).

In cases (b)–(k) every two $R_\mathfrak{m}$-orbits are CR-equivalent (and equivalent to $\mathfrak{m}$) by means of an automorphism of $M_\mathfrak{m}$ of a simple form specified below:

$$
\begin{aligned}
&\text{(b)} \quad [(z,w)] \mapsto [(az,aw)],\ a>0;\\
&\text{(c)} \quad z \mapsto z,\ w \mapsto w+a,\ a \in \mathbb{R};\\
&\text{(d)} \quad z \mapsto az,\ w \mapsto w,\ a>0;\\
&\text{(e)} \quad \text{as in (d)};\\
&\text{(f)} \quad z \mapsto z,\ w \mapsto aw,\ a>0;\\
&\text{(g)} \quad \text{as in (f)};\\
&\text{(h)} \quad z \mapsto az,\ w \mapsto aw,\ a>0;\\
&\text{(j)} \quad \text{as in (h)};\\
&\text{(j')} \quad z \mapsto z+a,\ w \mapsto e^a w,\ a \in \mathbb{R};\\
&\text{(j")} \quad z \mapsto a\,\mathrm{Re}\,z + ia^n\,\mathrm{Im}\,z,\ w \mapsto a\,\mathrm{Re}\,w + ia^n\,\mathrm{Im}\,w,\ a>0;\\
&\text{(k)} \quad \text{as in (h)}.
\end{aligned}
\tag{5.84}
$$

We will now show how strongly pseudoconvex orbits can be joined together to form (2,3)-manifolds. Our orbit gluing procedure is similar to that introduced in Sect. 3.4, and below we only indicate differences specific to the present case.

At step (II), if the group $G(M)$ is compact (in which case $\mathfrak{m}$ is one of $\mathcal{L}_m$ with $m \geq 2$, μ_α, $\mu_\alpha^{(2)}$, $\mu_\alpha^{(4)}$ with $\alpha > 1$), then every neighborhood of $O(p)$ contains a $G(M)$-invariant neighborhood. In this case, we extend f biholomorphically to some neighborhood of $O(p)$ and choose a $G(M)$-invariant neighborhood in it. We now assume that $G(M)$ is non-compact. In this case, as before, we extend the inverse map $\mathfrak{F} := f^{-1}$ to a map from an $R_\mathfrak{m}$-invariant neighborhood U' of $\mathfrak{m}$ onto a $G(M)$-invariant neighborhood W' of $O(p)$ in M. We will now show that the extended map is 1-to-1 on an $R_\mathfrak{m}$-invariant neighborhood of $\mathfrak{m}$ contained in U'.

Suppose that for some $s_0, s_0' \in U'$, $s_0 \neq s_0'$, we have $\mathfrak{F}(s_0) = \mathfrak{F}(s_0')$. This can only occur if s_0 and s_0' lie in the same $R_\mathfrak{m}$-orbit, and therefore there exist a point $s \in U$ and elements $h, h' \in R_\mathfrak{m}$ such that $s_0 = hs$, $s_0' = h's$. Then $h'^{-1}h \notin J_s$ and $\varphi^{-1}(h'^{-1}h) \in I_{\mathfrak{F}(s)}$, where J_s denotes the isotropy subgroup of s under the $R_\mathfrak{m}$-action. Since $\varphi^{-1}(J_s) \subset I_{\mathfrak{F}(s)}$, the group $I_{\mathfrak{F}(s)}$ contains more points than J_s.

Assume first that $\mathcal{O}(s)$ is non-spherical. It follows from the explicit forms of the models and the corresponding groups (see Proposition 5.3) that if for two locally CR-equivalent non-spherical models $\mathfrak{m}_1$, $\mathfrak{m}_2$ the groups $R_{\mathfrak{m}_1}$ and $R_{\mathfrak{m}_2}$ are isomorphic and the isotropy subgroup of a point in $\mathfrak{m}_1$ contains more points than that of a point in $\mathfrak{m}_2$, then $\mathfrak{m}_1 = \eta_\alpha^{(n)}$ and $\mathfrak{m}_2 = \eta_\alpha^{(2n)}$ for some α and odd $n \geq 1$ (here we set $\eta_\alpha^{(1)} := \eta_\alpha$). Hence $\mathcal{O}(s) = \eta_\alpha^{(2n)}$, and the model for $O(\mathfrak{F}(s))$ is $\eta_\alpha^{(n)}$ for some α and odd $n \geq 1$; consequently, $\mathfrak{m} = \eta_\beta^{(2n)}$ for some β. If there is a neighborhood of p not containing a point q such that the model for $O(q)$ is some $\eta_\gamma^{(n)}$, then $\mathfrak{F}$ is biholomorphic on an $R_\mathfrak{m}$-invariant open subset of U' containing $\mathfrak{m}$. Suppose now that in every neighborhood of p (that we assume to be contained in W) there is a point q such that $\mathcal{O}(f(q)) = \eta_\gamma^{(2n)}$ and the model for $O(q)$ is $\eta_\gamma^{(n)}$ for some γ. Note that I_q consists of two elements and $J_{f(q)}$ is trivial. Choose a sequence of such points $\{q_j\} \subset W$ converging to p. Let g_j be the non-trivial element of I_{q_j}. Since the action of $G(M)$ on M is proper and I_p is trivial, the sequence $\{g_j\}$ converges to the identity in $G(M)$. At the same time, the sequence $\{f(q_j)\}$ converges to $f(p)$ and therefore $\varphi(g_j)f(q_j)$ lies in U for large j. For large j we have $\mathfrak{F}(\varphi(g_j)f(q_j)) = q_j$. Since $\varphi(g_j)$ is a non-trivial element in $R_\mathfrak{m}$, the point $\varphi(g_j)f(q_j)$ does not coincide with $f(q_j)$. Thus, we have found two distinct points in U (namely, $f(q_j)$ and $\varphi(g_j)f(q_j)$ for large j) mapped by $\mathfrak{F}$ into the same point in W, which contradicts the fact that $\mathfrak{F}$ is 1-to-1 on U.

Assume now that $\mathcal{O}(s)$ is spherical. It follows from the explicit forms of the models and the corresponding groups (see Proposition 5.3) that if for two spherical models $\mathfrak{m}_1$, $\mathfrak{m}_2$ the groups $R_{\mathfrak{m}_1}$ and $R_{\mathfrak{m}_2}$ are isomorphic and the isotropy subgroup of a point in $\mathfrak{m}_1$ contains more points than that of a point in $\mathfrak{m}_2$, then $\mathfrak{m}_1 = \varepsilon'_{m/k_1}$, $R_{\mathfrak{m}_1} = \mathfrak{V}_{m/k_1}$, and $\mathfrak{m}_2 = \varepsilon'_{m/k_2}$, $R_{\mathfrak{m}_2} = \mathfrak{V}_{m/k_2}$, where $m, k_1, k_2 \in \mathbb{N}$, $(m, k_1) = 1$, $(m, k_2) = 1$, $k_1 > k_2$. Hence $\mathfrak{m} = \varepsilon'_{m/k_2}$, $\mathcal{O}(s)$ is equivalent to ε'_{m/k_2} by means of a map of the form (d) on list (5.84), and the model for $O(\mathfrak{F}(s))$ is ε'_{m/k_1} for some m, k_1, k_2 as above. If there is a neighborhood of p not containing a point q such that the model for $O(q)$ is some $\varepsilon'_{m/k}$, with $k \in \mathbb{N}$, $(m,k) = 1$, $k > k_2$, then $\mathfrak{F}$ is biholomorphic on an $R_{\mathfrak{m}}$-invariant open subset of U'. Suppose now that in every neighborhood of p (that we assume to be contained in W) there is a point q such that the model for $O(q)$ is $\varepsilon'_{m/k}$, with $k \in \mathbb{N}$, $(m,k) = 1$, $k > k_2$. Choose a sequence of such points $\{q_j\} \subset W$ converging to p. Since the action of $G(M)$ on M is proper, the isotropy subgroups I_{q_j} converge to I_p. Every subgroup I_{q_j} contains more points than $J_{f(q_{j_k})}$, and therefore for a subsequence $\{j_k\}$ of the sequence of indices $\{j\}$ there is a sequence of elements $\{g_{j_k}\}$ with $g_{j_k} \in I_{q_{j_k}}$, $\varphi(g_{j_k}) \notin J_{f(q_{j_k})}$ and such that $\{g_{j_k}\}$ converges to an element of I_p. Arguing now as in the non-spherical case, we obtain a contradiction with the fact that $\mathfrak{F}$ is 1-to-1 on U. Hence we have shown, as before, that f can be extended to a biholomorphic map satisfying (3.16) between a $G(M)$-invariant neighborhood of $O(p)$ in M and a $R_{\mathfrak{m}}$-invariant neighborhood of $\mathfrak{m}$ in $M_{\mathfrak{m}}$.

Next, at step (III) the map F establishes CR-equivalence between $\mathfrak{m}'$ and $\mathfrak{m}'_1$. Therefore, $\mathfrak{m}_1$ lies in $M_{\mathfrak{m}}$, that is, we have $M_{\mathfrak{m}} = M_{\mathfrak{m}_1}$. Moreover, F is either an element of $\mathrm{Aut}_{CR}(\mathfrak{m}')$ (if $\mathfrak{m}' = \mathfrak{m}'_1$), or is a composition of an element of $\mathrm{Aut}_{CR}(\mathfrak{m}')$ and a non-trivial map from list (5.84) that takes $\mathfrak{m}'$ onto $\mathfrak{m}'_1$ (if $\mathfrak{m}' \neq \mathfrak{m}'_1$); the latter is only possible in cases (b)–(k). It now follows from the explicit forms of CR-automorphisms of the models and the maps on list (5.84) that F extends to a holomorphic automorphism of $M_{\mathfrak{m}}$.

Further, if at step (IV) we have $V_j \subset D$ for $j = 1, 2$, then the map $\hat{F}$ establishes CR-equivalence between $\hat{\mathfrak{m}}_1$ and $\hat{\mathfrak{m}}_2$, where $\hat{\mathfrak{m}}_1 \neq \hat{\mathfrak{m}}_2$, which is only possible in cases (b)–(k). Cases (b), (c) and (f) can be dealt with exactly as cases (a), (b), (c) considered in Sect. 3.4. The same applies to cases (d), (e), provided the group $R_{\mathfrak{m}}$ is either $\mathfrak{V}_\alpha$ or $\mathfrak{V}_\infty$, respectively. Furthermore, if in cases (d) and (e) the group $R_{\mathfrak{m}}$ is given in terms of the group $\mathcal{T}$ (see (5.83)), then, using the fact that $\hat{F}$ is $R_{\mathfrak{m}}$-equivariant, we again obtain that $\hat{F}$ is either a dilation or translation in z_n, respectively. Next, in cases (g)–(k) the $R_{\mathfrak{m}}$-equivariance of $\hat{F}$ implies that $\hat{F} = \mathrm{id}$, which is impossible. Thus, in all cases we obtain a contradiction, and hence M is holomorphically equivalent to an $R_{\mathfrak{m}}$-invariant domain in $M_{\mathfrak{m}}$.

All hyperbolic $R_{\mathfrak{m}}$-invariant domains in each of cases (b)-(n") are described as follows:

(b) $S_{m,s,t} := \left\{(z,w) \in \mathbb{C}^2 : s < |z|^2 + |w|^2 < t\right\}/\mathbb{Z}_m$,
$0 \le s < t < \infty$ (cf. (3.19)); (5.85)

(c) $\left\{(z,w) \in \mathbb{C}^2 : s + |z|^2 < \mathrm{Re}\, w < t + |z|^2\right\}$, $-\infty < s < t \le \infty$ (cf. (3.20));

(d) $\mathfrak{R}_{\alpha,s,t} := \{(z,w) \in \mathbb{C}^2 : s|w|^\alpha < \operatorname{Re} z < t|w|^\alpha,\ w \neq 0\}$,
$0 \leq s < t \leq \infty$, where $s = 0$ and $t = \infty$ do not hold simultaneously (cf. (3.23)); (5.86)

(e) $\mathfrak{R}_{s,t} := \{(z,w) \in \mathbb{C}^2 : s \exp(\operatorname{Re} w) < \operatorname{Re} z < t \exp(\operatorname{Re} w)\}$,
$0 \leq s < t \leq \infty$, where $s = 0$ and $t = \infty$ do not hold simultaneously (cf. (3.22));

(f) $\{(z,w) \in \mathbb{C}^2 : s \exp(|z|^2) < |w| < t \exp(|z|^2)\}$, $0 < s < t \leq \infty$ (cf. (3.21));

(g) $R_{\alpha,s,t}$, $0 \leq s < t \leq \infty$, where $s = 0$ and $t = \infty$ do not hold simultaneously (see (5.1));

(h) $U_{s,t}$, $0 \leq s < t \leq \infty$, where $s = 0$ and $t = \infty$ do not hold simultaneously (see (5.6));

(j) $\mathfrak{S}_{s,t}$, $0 \leq s < t < \infty$ (see (5.9));

(j') $\mathfrak{S}^{(\infty)}_{s,t}$, $0 \leq s < t < \infty$ (see (5.15));

(j") $\mathfrak{S}^{(n)}_{s,t}$, $0 \leq s < t < \infty$, $n \geq 2$ (see (5.15));

(k) $V_{\alpha,s,t}$, $0 < t < \infty$, $e^{-2\pi\alpha}t < s < t$ (see (5.20));

(l) $E_{s,t}$, $1 \leq s < t < \infty$ (see (5.23));

(l') $E^{(4)}_{s,t}$, $1 \leq s < t < \infty$ (see (5.30));

(l") $E^{(2)}_{s,t}$, $1 \leq s < t < \infty$ (see (5.30));

(m) $\Omega_{s,t}$, $-1 \leq s < t \leq 1$ (see (5.37));

(m') $\Omega^{(\infty)}_{s,t}$, $-1 \leq s < t \leq 1$ (see (5.55));

(m") $\Omega^{(n)}_{s,t}$, $-1 \leq s < t \leq 1$, $n \geq 2$ (see (5.55));

(n) $D_{s,t}$, $1 \leq s < t \leq \infty$ (see (5.43));

(n') $D^{(\infty)}_{s,t}$, $1 \leq s < t \leq \infty$ (see (5.60));

(n") $D^{(2)}_{s,t}$, $1 \leq s < t \leq \infty$ (see (5.60));

(n") $D^{(2n)}_{s,t}$, $1 \leq s < t \leq \infty$, $n \geq 2$ (see (5.60));

(n") $D^{(n)}_{s,t}$, $1 \leq s < t \leq \infty$, $n \geq 3$, n is odd (see **(10)**).

This concludes our orbit gluing procedure. Note that in each of cases (d) and (e) we have two non-isomorphic possibilities for $R_{\mathfrak{m}}$. Each of the possibilities leads to the same set of $R_{\mathfrak{m}}$-invariant domains.

We now observe that the automorphism groups of all $R_{\mathfrak{m}}$-invariant domains that appear in cases (b)–(f) have dimension at least 4. Finally, excluding equivalent domains leads to list (i)–(xv) as stated in the theorem.

The proof of Theorem 5.2 is complete. ∎

5.3 Levi-Flat Orbits

In this section we give a classification of (2,3)-manifolds M for which every $G(M)$-orbit has codimension 1 and at least one orbit is Levi-flat. We start by classifying all possible Levi-flat orbits up to CR-diffeomorphisms together with group actions.

Observe, first of all, that the Levi-flat hypersurface $\mathcal{O}_1$ (see (5.4)) is an orbit of the action of each of the groups G_α (see (5.2)) for $\alpha \in \mathbb{R}$ (including $\alpha = 0$) and $\mathfrak{G}$ (see (5.7)) on $\mathbb{C}^2$. Recall next that the Levi-flat hypersurface $\mathcal{O}_0^{(n)}$ (see (5.71), (5.75)) is an orbit of the action of $\mathcal{R}^{(n)}$ (see **(10)**, (5.56)) on $\mathcal{Q}_-$ for $n = 1$ (see (5.48)) and on $M^{(n)}$ for $n \in \mathbb{N}$, $n \geq 2$ (see **(11)(d)**). Furthermore, the Levi-flat hypersurface $\mathcal{O}_0^{(\infty)}$ (see (5.76)) is an orbit of the action of $\mathcal{R}^{(\infty)}$ (see (5.58)) on $M^{(\infty)}$ (see **(11)(d)**). Note also that the Levi-flat hypersurface

$$\mathcal{O}_1' := \left\{(z,w) \in \mathbb{C}^2 : \operatorname{Re} z > 0,\ |w| = 1\right\} \tag{5.87}$$

is an orbit of the action on $\mathbb{C}^2$ of the group G_0' of all maps

$$\begin{aligned} z &\mapsto \lambda z + i\beta,\\ w &\mapsto e^{i\psi} w, \end{aligned} \tag{5.88}$$

where $\lambda > 0$, $\beta, \psi \in \mathbb{R}$. The hypersurface $\mathcal{O}_0^{(\infty)}$ is CR-equivalent to $\mathcal{O}_1$, and the hypersurface $\mathcal{O}_0^{(n)}$ is CR-equivalent to $\mathcal{O}_1'$ for every $n \in \mathbb{N}$.

We will now prove the following proposition. Note that it applies to (2,3)-manifolds possibly containing codimension 2 orbits.

Proposition 5.4. ([I5]) *Let M be a (2,3)-manifold. Assume that for a point $p \in M$ its orbit $O(p)$ is Levi-flat. Then one of the following holds:*

(i) $O(p)$ is equivalent to $\mathcal{O}_1$ by means of a real-analytic CR-map that transforms $G(M)|_{O(p)}$ into either the group $G_\alpha|_{\mathcal{O}_1}$ for some $\alpha \in \mathbb{R}$ or the group $\mathfrak{G}|_{\mathcal{O}_1}$;
(ii) $O(p)$ is equivalent to $\mathcal{O}_1'$ by means of a real-analytic CR-map that transforms $G(M)|_{O(p)}$ into the group $G_0'|_{\mathcal{O}_1'}$;
(iii) $O(p)$ is equivalent to $\mathcal{O}_0^{(j)}$ for some $j \in \{1, 2, \dots, \infty\}$ by means of a real-analytic CR-map that transforms $G(M)|_{O(p)}$ into the group $\mathcal{R}^{(j)}|_{\mathcal{O}_0^{(j)}}$.

Proof: Recall that the hypersurface $O(p)$ is foliated by complex submanifolds equivalent to Δ (see (ii) of Proposition 4.2). For convenience, we realize Δ as the right half-plane $\mathcal{P} = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. We begin as in Case 2 of the proof of Proposition 4.2 (see also the proof of Proposition 3.5). Denote, as before, by $\mathfrak{g}(M)$ the Lie algebra of $G(M)$-vector fields on M and identify it with the Lie algebra of $G(M)$. Further, we identify every vector field from $\mathfrak{g}(M)$ with its restriction to $O(p)$. For $q \in O(p)$ we consider the leaf $O(p)_q$ of the foliation passing through q and the subspace $\mathfrak{l}_q \subset \mathfrak{g}(M)$ of all vector fields tangent to $O(p)_q$ at q. Since vector fields in $\mathfrak{l}_q$ remain tangent to $O(p)_q$ at each point in $O(p)_q$, the subspace $\mathfrak{l}_q$ is in fact a Lie subalgebra of $\mathfrak{g}(M)$. It follows from the definition of $\mathfrak{l}_q$ that $\dim \mathfrak{l}_q = 2$.

Denote by H_q the (possibly non-closed) connected subgroup of $G(M)$ with Lie algebra $\mathfrak{l}_q$. It is straightforward to verify that the group H_q acts on $O(p)_q$ by holomorphic transformations. If some element $g \in H_q$ acts trivially on $O(p)_q$, then $g \in I_q$. If for every non-identical element of L_q its projection to the U_1-component (see (ii) of Proposition 4.2) is non-identical, then every non-identical element of I_q acts non-trivially on $O(p)_q$ and thus $g = \mathrm{id}$; if L_q contains a non-identical element with an identical projection to the U_1-component and $g \neq \mathrm{id}$, then $g = g_q$, where, as in the proof of Proposition 4.2, g_q denotes the element of I_q corresponding to the non-trivial element in $\mathbb{Z}_2$ (see (ii) of Proposition 4.2). Thus, $\dim H_q = 2$, and either H_q or $H_q/\mathbb{Z}_2$ acts effectively on $O(p)_q$ (the former case occurs if $g_q \notin H_q$, the latter if $g_q \in H_q$). As we noted in the proof of Proposition 5.3, every 2-dimensional (a priori not necessarily closed) subgroup of $\operatorname{Aut}(\mathcal{P})$ is conjugate in $\operatorname{Aut}(\mathcal{P})$ to the subgroup $\mathcal{T}$ (see (5.83)). The Lie algebra of this subgroup is isomorphic to the 2-dimensional Lie algebra $\mathfrak{h}$ given by two generators X and Y satisfying $[X, Y] = X$. Therefore, $\mathfrak{l}_q$ is isomorphic to $\mathfrak{h}$ for every $q \in O(p)$.

It is straightforward to determine all 3-dimensional Lie algebras containing a subalgebra isomorphic to $\mathfrak{h}$. Every such algebra has generators X, Y, Z that satisfy one of the following sets of relations:

$$\begin{aligned} &\text{(R1) } [X,Y] = X,\ [Z,X] = 0,\ [Z,Y] = \alpha Z, \qquad \alpha \in \mathbb{R},\\ &\text{(R2) } [X,Y] = X,\ [Z,X] = 0,\ [Z,Y] = X + Z,\\ &\text{(R3) } [X,Y] = X,\ [Z,X] = Y,\ [Z,Y] = -Z. \end{aligned} \tag{5.89}$$

Suppose first that $\mathfrak{g}(M)$ is given by relations (R1). In this case $\mathfrak{g}(M)$ is isomorphic to the Lie algebra of the simply-connected Lie group G_α (see (5.2)).

Indeed, the Lie algebra of G_α is isomorphic to the Lie algebra of vector fields on $\mathbb{C}^2$ with the generators

$$\begin{aligned} X_1 &:= i\,\partial/\partial z,\\ Y_1 &:= z\,\partial/\partial z + \alpha w\,\partial/\partial w,\\ Z_1 &:= i\,\partial/\partial w, \end{aligned}$$

that clearly satisfy (R1).

Assume first that $\alpha \neq 0$. In this case the center of G_α is trivial, and hence G_α is the only (up to isomorphism) connected Lie group whose Lie algebra is given by relations (R1). Therefore, $G(M)$ is isomorphic to G_α. Assume further that $\alpha \neq 1$. In this case, it is straightforward to observe that every subalgebra of $\mathfrak{g}(M)$ isomorphic to $\mathfrak{h}$ is generated either by X_1 and $Y_1+\nu Z_1$, or by Z_1 and $\nu X_1 + Y_1$ for some $\nu \in \mathbb{R}$. The connected subgroup of G_α with Lie algebra generated by X_1 and $Y_1+\nu Z_1$ is conjugate in G_α to the closed subgroup H^1_α given by $\gamma = 0$ in (5.2); similarly, the connected subgroup of G_α with Lie algebra generated by Z_1 and $\nu X_1 + Y_1$ is conjugate to the closed subgroup H^2_α given by $\beta = 0$ in (5.2). Moreover, the conjugating element can be chosen to belong to the subgroup $\mathcal{W}^1$ of maps of the form

$$\begin{aligned} z &\mapsto z\\ w &\mapsto w + i\gamma, \quad \gamma \in \mathbb{R}, \end{aligned} \tag{5.90}$$

in the first case, and to the subgroup $\mathcal{W}^2$ of maps of the form

$$\begin{aligned} z &\mapsto z + i\beta, \quad \beta \in \mathbb{R},\\ w &\mapsto w, \end{aligned}$$

in the second case. These subgroups are one-parameter subgroups of G_α arising from Z_1 and X_1, respectively.

Thus, upon identifying $G(M)$ with G_α, the subgroup H_q for every $q \in O(p)$ is conjugate to either H^1_α or H^2_α by an an element of either $\mathcal{W}^1$ or $\mathcal{W}^2$, respectively. In particular, H_q is isomorphic to $\mathcal{T}$ and hence does not have subgroups isomorphic to $\mathbb{Z}_2$. Therefore, H_q acts effectively on $O(p)_q$. Since the subgroups H_q are conjugate to each other, it follows that either H_q is conjugate to H^1_α for every q, or H_q is conjugate to H^2_α for every q. Suppose first that the former holds. Then for every $q \in O(p)$ every element of $G(M)$ can be written as gh, where $g \in \mathcal{W}^1$, $h \in H_q$. Hence for every $q_1, q_2 \in O(p)$ there exists $g \in \mathcal{W}^1$ such that $gO(p)_{q_1} = O(p)_{q_2}$. Furthermore, since the normalizer of H^1_α in G_α coincides with H^1_α, such an element g is unique. Let $q_0 \in O(p)$ be a point for which $H_{q_0} = H^1_\alpha$, and let $f : O(p)_{q_0} \to \mathcal{P}$ be a holomorphic equivalence that transforms $H_{q_0}|_{O(p)_{q_0}}$ into the group $\mathcal{T}$. Let $\hat{X}_1$ and $\hat{Y}_1$ be the vector fields on $O(p)$ corresponding to X_1, Y_1. Under the map f the vector fields $\hat{X}_1|_{O(p)_{q_0}}$ and $\hat{Y}_1|_{O(p)_{q_0}}$ (which are tangent to $O(p)_{q_0}$) transform into some vector fields X_1^* and Y_1^* on $\mathcal{P}$ such that $[X_1^*, Y_1^*] = X_1^*$. Clearly, X_1^* and Y_1^* generate the algebra of $\mathcal{T}$-vector fields on $\mathcal{P}$. It is straightforward to verify that one can find an element of $\mathcal{T}$ that transforms Y_1^* into $z\,\partial/\partial z = Y_1|_{\mathcal{P}}$ and X_1^* into one of $\pm i\,\partial/\partial z = \pm X_1|_{\mathcal{P}}$, and therefore we can assume that f is chosen so that it transforms $\hat{X}_1|_{O(p)_{q_0}}$ and $\hat{Y}_1|_{O(p)_{q_0}}$ into $\pm X_1|_{\mathcal{P}}$, $Y_1|_{\mathcal{P}}$, respectively.

For every $q \in O(p)$ we now find the unique element $g \in \mathcal{W}^1$ such that $gO(p)_{q_0} = O(p)_q$ and define $F(q) := \Big(f(g^{-1}(q)), i\gamma\Big) \in \mathbb{C}^2$, with γ corresponding to g as in formula (5.90). Clearly, F is a real-analytic CR-isomorphism between $O(p)$ and $\mathcal{O}_1$ that transforms $\hat{Z}_1$ into $i\,\partial/\partial w|_{\mathcal{O}_1} = Z_1|_{\mathcal{O}_1}$, where $\hat{Z}_1$ is the vector field on $O(p)$ corresponding to Z_1 (recall that $\mathcal{W}^1 = \{\exp(sZ_1), s \in \mathbb{R}\}$).

Denote by $\tilde{X}$, $\tilde{Y}$ the vector fields on $\mathcal{O}_1$ into which F transforms $\hat{X}_1$, $\hat{Y}_1$, respectively. Since F is real-analytic, it extends to a biholomorphic map from a neighborhood of $O(p)$ in M onto a neighborhood of $\mathcal{O}_1$ in $\mathbb{C}^2$. Clearly, $\hat{X}_1$, $\hat{Y}_1$, extend to holomorphic vector fields on all of M and hence $\tilde{X}$, $\tilde{Y}$, extend to holomorphic vector fields defined in a neighborhood of $\mathcal{O}_1$. Since the restrictions of $\tilde{X}$ and $\tilde{Y}$ to $\mathcal{P} \times \{0\} \subset \mathcal{O}_1$ are $\pm\, i\partial/\partial z$ and $z\,\partial/\partial z$, respectively, these vector fields have the forms

$$\tilde{X} = (\pm i + \rho(z,w))\partial/\partial z + \sigma(z,w)\partial/\partial w, \tag{5.91}$$

and

$$\tilde{Y} = (z + \mu(z,w))\partial/\partial z + \tau(z,w)\partial/\partial w, \tag{5.92}$$

where ρ, σ, μ, τ are functions holomorphic near $\mathcal{O}_1$ and such that

$$\rho(z,0) \equiv \sigma(z,0) \equiv \mu(z,0) \equiv \tau(z,0) \equiv 0. \tag{5.93}$$

Since $[\hat{Z}_1, \hat{X}_1] = 0$ and $[\hat{Z}_1, \hat{Y}_1] = \alpha \hat{Z}_1$ on $O(p)$, on a neighborhood of $\mathcal{O}_1$ we obtain

$$[Z_1, \tilde{X}] = 0,\ [Z_1, \tilde{Y}] = \alpha Z_1. \tag{5.94}$$

Conditions (5.93), (5.94) imply: $\rho \equiv 0$, $\sigma \equiv 0$, $\mu \equiv 0$, $\tau = \alpha w$. Thus, $\tilde{X} = \pm X_1$, $\tilde{Y} = Y_1$, and hence F transforms $G(M)|_{O(p)}$ into $G_\alpha|_{\mathcal{O}_1}$.

The case when H_q is conjugate to H_α^2 for every $q \in O(p)$ is treated similarly; arguing as above we construct a real-analytic CR-isomorphism between $O(p)$ and $\hat{\mathcal{O}}_1$ (see (5.5)) that transforms $G(M)|_{O(p)}$ into $G_\alpha|_{\hat{\mathcal{O}}_1}$. Further, interchanging the variables turns $\hat{\mathcal{O}}_1$ into $\mathcal{O}_1$ and G_α into $G_{1/\alpha}$.

Suppose now that $\alpha = 1$. In this case, in addition to the subalgebras arising for $\alpha \neq 1$, a subalgebra of $\mathfrak{g}(M)$ isomorphic to $\mathfrak{h}$ can also be generated by $X_1 + \eta Z_1$ and $Y_1 + \nu Z_1$ for some $\eta, \nu \in \mathbb{R}$, $\eta \neq 0$. The connected subgroup of G_1 corresponding to this subalgebra is conjugate in G_1 to the closed subgroup $H_{1,\eta}$ of all maps of the form (5.2) with $\alpha = 1$, $\gamma = \beta\eta$. Moreover, the conjugating element can be chosen to belong to the subgroup $\mathcal{W}^1$ (see (5.90)). Thus, upon identifying $G(M)$ with G_1, the subgroup H_q for every $q \in O(p)$ is conjugate to either H_1^1 or H_1^2, or $H_{1,\eta}$ for some $\eta \neq 0$ (all these subgroups are closed). In particular, H_q is isomorphic to $\mathcal{T}$ and hence acts effectively on $O(p)_q$. Since the subgroups H_q are conjugate to each other, it follows that either H_q is conjugate to H_1^1 for every q, or H_q is conjugate to H_1^2 for every q, or H_q is conjugate to $H_{1,\eta}$ for every q and a fixed η. The first two cases are treated as for $\alpha \neq 1$. Suppose that H_q is conjugate to $H_{1,\eta}$ for every $q \in O(p)$. It can be shown, as before, that for every $q_1, q_2 \in O(p)$ there exists a unique $g \in \mathcal{W}^1$ such that $gO(p)_{q_1} = O(p)_{q_2}$. Fix $q_0 \in O(p)$ with the property $H_{q_0} = H_{1,\eta}$, and let $f : O(p)_{q_0} \to \mathcal{P}$, with be a holomorphic equivalence that transforms $H_{q_0}|_{O(p)_{q_0}}$ into the group $\mathcal{T}$ and such that $\hat{X}_1 + \eta \hat{Z}_1|_{O(p)_{q_0}}$ and $\hat{Y}_1|_{O(p)_{q_0}}$ (which are tangent to $O(p)_{q_0}$) are transformed into the vector fields $\pm X_1|_{\mathcal{P}}$ and $Y_1|_{\mathcal{P}}$, respectively. For every $q \in O(p)$ we now find the unique map $g \in \mathcal{W}^1$ such that $gO(p)_{q_0} = O(p)_q$ and define $F(q) := \Big(f(g^{-1}(q)), i\gamma\Big)$, with γ corresponding to g as in formula (5.90). Analogously to the case $\alpha \neq 1$ we obtain: $\tilde{X} = \pm X_1 - \eta Z_1$, $\tilde{Y} = Y_1$. Hence F transforms $G(M)|_{O(p)}$ into $G_1|_{\mathcal{O}_1}$.

Suppose now that $\alpha = 0$. In this case there are exactly two (up to isomorphism) connected Lie groups with Lie algebra $\mathfrak{g}(M)$: G_0 and G_0' (see (5.88)). It is straightforward to see that every subalgebra of $\mathfrak{g}(M)$ isomorphic to $\mathfrak{h}$ is generated by X_1 and $Y_1 + \nu Z_1$ for some $\nu \in \mathbb{R}$. Clearly, the connected subgroup of G_0 with Lie algebra generated by X_1 and $Y_1 + \nu Z_1$ coincides with the closed normal subgroup $H_{0,\nu}$ given by $\lambda = e^t$, $\gamma = \nu t$, $t \in \mathbb{R}$ (see (5.2)). It then follows that if $G(M)$ is isomorphic to G_0, there exists $\nu \in \mathbb{R}$, such that, identifying $G(M)$ and G_0, we have $H_q = H_{0,\nu}$ for every $q \in O(p)$. Further, let us realize the Lie algebra of G_0' as the Lie algebra generated by the following vector fields on $\mathbb{C}^2$: X_1, Y_1, $Z_1' := iw\,\partial/\partial w$, which clearly satisfy (R1) of (5.89). The connected subgroup of G_0' with Lie algebra generated by X_1 and $Y_1 + \nu Z_1'$ coincides with the closed normal subgroup $H_{0,\nu}'$ of G_0' given by $\lambda = e^t$, $\psi = \nu t$, $t \in \mathbb{R}$ (see (5.88)). It then follows that if $G(M)$ is isomorphic to G_0', there exists $\nu \in \mathbb{R}$, such that, identifying $G(M)$ and G_0', we have $H_q = H_{0,\nu}'$ for every $q \in O(p)$.

Thus, if $\alpha = 0$, every subgroup H_q is normal, closed, isomorphic to $\mathcal{T}$ (hence acts effectively on $O(p)_q$). In particular, all these subgroups coincide for $q \in O(p)$. Denote by H the coinciding subgroups H_q. The group H acts properly on $O(p)$, and the orbits of this action are the leaves $O(p)_q$ of the foliation of $O(p)$. Further, we have $G(M) = H \times L$, where L is either the subgroup $\mathcal{W}^1$ (see (5.90)), or the subgroup $\mathcal{W}'^1$ given by $\lambda = 1$, $\beta = 0$ in formula (5.88), and hence is isomorphic to either $\mathbb{R}$ or S^1. For every $q \in O(p)$ let $S_q := \{g \in L : gO(p)_q = O(p)_q\}$. Since $O(p)_q$ is closed, S_q is a closed subgroup of L. Clearly, for every $g \in S_q$ there is $h \in H$ such that $hg \in I_q$. The elements g and h lie in the projections of I_q to L and H, respectively. Since H is isomorphic to $\mathcal{T}$, it does not have non-trivial finite subgroups, hence the projection of I_q to H is trivial, and therefore $S_q = I_q$. Since all isotropy subgroups are contained in the Abelian subgroup L and are conjugate to each other in $G(M)$, they are in fact identical. The effectiveness of the action of $G(M)$ on M now implies that all isotropy subgroups are trivial and hence every S_q is trivial as well.

Thus, we have shown that for every $q_1, q_2 \in O(p)$ there is a unique $g \in L$, such that $gO(p)_{q_1} = O(p)_{q_2}$. Suppose first that $L = \mathcal{W}^1$. Fix $q_0 \in O(p)$, and let $f : O(p)_{q_0} \to \mathcal{P}$ be a holomorphic equivalence that transforms $H|_{O(p)_{q_0}}$ into the group $\mathcal{T}$ and $\hat{X}_1|_{O(p)_{q_0}}, \hat{Y}_1 + \nu\hat{Z}_1|_{O(p)_{q_0}}$ into $\pm X_1|_{\mathcal{P}}, Y_1|_{\mathcal{P}}$, respectively. For every $q \in O(p)$ find the unique map $g \in \mathcal{W}^1$ such that $gO(p)_{q_0} = O(p)_q$ and define $F(q) := \Big(f(g^{-1}(q)), i\gamma\Big)$, with γ corresponding to g as in formula (5.90). It can now be shown as in the case $\alpha \neq 0$ that F transforms $G(M)|_{O(p)}$ into $G_0|_{\mathcal{O}_1}$.

Suppose now that $L = \mathcal{W'}^1$. Fix $q_0 \in O(p)$, and let $f : O(p)_{q_0} \to \mathcal{P}$ be a holomorphic equivalence that transforms $H|_{O(p)_{q_0}}$ into the group $\mathcal{T}$ and $\hat{X}_1|_{O(p)_{q_0}}, \hat{Y}_1 + \nu\hat{Z}_1'|_{O(p)_{q_0}}$ into $\pm X_1|_{\mathcal{P}}, Y_1|_{\mathcal{P}}$, respectively, where $\hat{Z}_1'$ denotes the vector field on $O(p)$ corresponding to Z_1'. For every $q \in O(p)$ find the unique map $g \in \mathcal{W'}^1$ such that $gO(p)_{q_0} = O(p)_q$ and define $F(q) := \Big(f(g^{-1}(q)), e^{i\psi}\Big)$, with $e^{i\psi}$ corresponding to g as in formula (5.88). Clearly, F is a real-analytic CR-isomorphism between $O(p)$ and $\mathcal{O}_1'$ (see (5.87)) that transforms $\hat{Z}_1'$ into $iw\,\partial/\partial w|_{\mathcal{O}_1'} = Z_1'|_{\mathcal{O}_1'}$.

As before, denote by $\tilde{X}$, $\tilde{Y}$ the vector fields on $\mathcal{O}_1$ into which F transforms $\hat{X}_1$, $\hat{Y}_1$, respectively. These vector fields extend to holomorphic vector fields defined in a neighborhood of $\mathcal{O}_1'$. Since the restrictions of $\tilde{X}$ and $\tilde{Y} + \nu Z_1'$ to $\mathcal{P} \times \{1\} \subset \mathcal{O}_1'$ are $\pm X_1|_{\mathcal{P}}$ and $Y_1|_{\mathcal{P}}$, these vector fields have the forms that appear in the right-hand sides of formulas (5.91), (5.92), respectively, where ρ, σ, μ, τ are functions holomorphic near $\mathcal{O}_1'$ and such that

$$\rho(z,1) \equiv \sigma(z,1) \equiv \mu(z,1) \equiv \tau(z,1) \equiv 0. \tag{5.95}$$

Since $[\hat{Z}_1', \hat{X}_1] = [\hat{Z}_1', \hat{Y}_1 + \nu\hat{Z}_1'] = 0$ on $O(p)$, on a neighborhood of $\mathcal{O}_1'$ we obtain

$$[Z_1', \tilde{X}] = [Z_1', \tilde{Y} + \nu Z_1'] = 0. \tag{5.96}$$

Conditions (5.95), (5.96) imply: $\rho \equiv \sigma \equiv \mu \equiv \tau \equiv 0$. Thus, $\tilde{X} = \pm X_1$, $\tilde{Y} = Y_1 - \nu Z_1'$, and hence F transforms $G(M)|_{O(p)}$ into $G_0'|_{\mathcal{O}_1'}$.

Suppose next that $\mathfrak{g}(M)$ is given by relations (R2) (see (5.89)). In this case $\mathfrak{g}(M)$ is isomorphic to the Lie algebra of the simply-connected Lie group $\mathfrak{G}$ (see (5.7)). Indeed, the Lie algebra of $\mathfrak{G}$ is isomorphic to the Lie algebra of holomorphic vector fields on $\mathbb{C}^2$ with the following generators: X_1, $Y_2 := (z+w)\,\partial/\partial z + w\,\partial/\partial w$, Z_1, which clearly satisfy (R2). It is straightforward to observe that the center of $\mathfrak{G}$ is trivial, and hence $\mathfrak{G}$ is the only (up to isomorphism) connected Lie group whose Lie algebra is given by relations (R2). Therefore, $G(M)$ is isomorphic to $\mathfrak{G}$. In this case every subalgebra of $\mathfrak{g}(M)$ isomorphic to $\mathfrak{h}$ is generated by X_1 and $Y_2 + \nu Z_1$ for some $\nu \in \mathbb{R}$. The connected subgroup of $\mathfrak{G}$ with Lie algebra generated by X_1 and $Y_2 + \nu Z_1$ is conjugate in $\mathfrak{G}$ to the closed subgroup Q given by $\gamma = 0$ (see (5.7)). Moreover, the conjugating element can be chosen to belong to $\mathcal{W}^1$ (see (5.90)).

Thus – upon identification of $G(M)$ and $\mathfrak{G}$ – the subgroup H_q for every $q \in O(p)$ is conjugate to Q by an element of $\mathcal{W}^1$. Further, since the normalizer of Q in $\mathfrak{G}$ coincides with Q, we proceed as in the case of the group G_α for $\alpha \neq 0$ and obtain that there exists a real-analytic CR-isomorphism F between $O(p)$ and $\mathcal{O}_1$ that transforms $\hat{Z}_1$ into $Z_1|_{\mathcal{O}_1}$ and the corresponding vector fields $\hat{X}_1$, $\hat{Y}_2$ on a neighborhood of $O(p)$ in M into holomorphic vector fields $\tilde{X}$, $\tilde{Y}$ of the forms appearing in the right-hand sides of formulas (5.91), (5.92), respectively, where ρ, σ, μ, τ are functions holomorphic near $\mathcal{O}_1$ and satisfying (5.93). Since $[\hat{Z}_1, \hat{X}_1] = 0$ and $[\hat{Z}_1, \hat{Y}_2] = \hat{X}_1 + \hat{Z}_1$ on $O(p)$, on a neighborhood of $\mathcal{O}_1$ we obtain

$$[Z_1, \tilde{X}] = 0, \; [Z_1, \tilde{Y}] = \tilde{X} + Z_1. \tag{5.97}$$

Conditions (5.93), (5.97) imply: $\rho \equiv 0$, $\sigma \equiv 0$, $\mu \equiv \pm w$, $\tau = w$, respectively. Thus, we have either $\tilde{X} = X_1$, $\tilde{Y} = Y_2$, or $\tilde{X} = -X_1$, $\tilde{Y} = (z - w)\, \partial/\partial z + w\, \partial/\partial w$. Hence either F or $S \circ F$ transforms $G(M)|_{O(p)}$ into $\mathfrak{G}|_{\mathcal{O}_1}$, where S is the map given by formula (5.81).

Suppose finally that $\mathfrak{g}(M)$ is given by relations (R3) (see (5.89)). In this case $\mathfrak{g}(M)$ is isomorphic to the algebra $\mathfrak{so}_{2,1}(\mathbb{R})$. All connected Lie groups with such Lie algebra are described as follows: any simply-connected group is isomorphic to the group $\mathfrak{V}_\infty$, and any non simply-connected group is isomorphic to $\mathfrak{V}_n$ with $n \in \mathbb{N}$, where $\mathfrak{V}_\infty$ and $\mathfrak{V}_n$ are the Lie groups defined in Proposition 5.3. Clearly, the set $\mathcal{C}' := \big\{(z, w) \in \mathbb{C}^2 : \operatorname{Re} z > 0\big\}$ is $\mathfrak{V}_j$-invariant for $j \in \{1, 2, \ldots, \infty\}$.

Consider in $\mathfrak{V}_j$ three one-parameter subgroups of transformations of $\mathcal{C}'$

for $j = \infty$:

$$\begin{aligned} &z \mapsto z - \frac{i}{2}\beta, && w \mapsto w,\\ &z \mapsto \lambda z, && w \mapsto w + \ln_0 \lambda,\\ &z \mapsto \frac{z}{i\mu z + 1}, && w \mapsto w - 2\ln_0(i\mu z + 1), \end{aligned}$$

for $j = n \in \mathbb{N}$:

$$\begin{aligned} &z \mapsto z - \frac{i}{2}\beta, && w \mapsto w,\\ &z \mapsto \lambda z, && w \mapsto \lambda^{1/n} w,\\ &z \mapsto \frac{z}{i\mu z + 1}, && w \mapsto \frac{1}{(i\mu z + 1)^{2/n}} w. \end{aligned}$$

where $\lambda > 0$, $\beta, \mu \in \mathbb{R}$, $t^{2/n} = \exp(2/n \ln_0 t)$ for $t \in \mathbb{C} \setminus (-\infty, 0]$, and $\ln_0$ is the branch of the logarithm in $\mathbb{C} \setminus (-\infty, 0]$ defined by the condition $\ln_0 1 = 0$. The vector fields corresponding to these subgroups generate the Lie algebras of $\mathfrak{V}_j$ for $j \in \{1, 2, \ldots, \infty\}$ and are as follows:

for $j = \infty$:

$$\begin{aligned} X_3 &:= -\frac{i}{2}\, \partial/\partial z,\\ Y_3 &:= z\, \partial/\partial z + \partial/\partial w,\\ Z_3 &:= -iz^2\, \partial/\partial z - 2iz\, \partial/\partial w, \end{aligned}$$

for $j = n \in \mathbb{N}$:

$$X_3 := -\frac{i}{2}\,\partial/\partial z,$$

$$Y_3 := z\,\partial/\partial z + \frac{w}{n}\,\partial/\partial w,$$

$$Z_3 := -iz^2\,\partial/\partial z - \frac{2izw}{n}\,\partial/\partial w.$$

One can verify that these vector fields indeed satisfy relations (R3).

Next, it is straightforward to show that any subalgebra of $\mathfrak{g}(M)$ isomorphic to $\mathfrak{h}$ is generated by either $X_3 + \eta Y_3 - \eta^2/2\, Z_3$, $Y_3 - \eta Z_3$, with $\eta \in \mathbb{R}$, or by Y_3, Z_3. For every $j \in \{1, 2, \dots, \infty\}$ the connected subgroups of $\mathfrak{V}_j$ corresponding to the subalgebras generated by $X_3 + \eta Y_3 - \eta^2/2\, Z_3$, $Y_3 - \eta Z_3$, with $\eta \in \mathbb{R}$, or by Y_3, Z_3 are isomorphic to $\mathcal{T}$ (hence H_q acts effectively on $O(p)_q$ for every q) and are all closed and conjugate to each other in $\mathfrak{V}_j$ by elements of the one-parameter subgroup of $\mathfrak{V}_j$ arising from $X_3 - 1/2\, Z_3$. We denote this subgroup by $\mathcal{W}_j$ and describe it in more detail. Let first $j = \infty$. Transform $\mathcal{C}'$ into $\mathcal{C} = \{(z, w) : |z| < 1\}$ (cf. Sect. 3.4) by means of the map

$$z \mapsto \frac{z-1}{z+1},$$

$$w \mapsto w - 2\ln_0\left((z+1)/2\right).$$

Then $\mathcal{W}_\infty$ is the subgroup of $\mathfrak{V}_\infty$ that transforms into the group of maps

$$\begin{aligned} z &\mapsto e^{it} z, \\ w &\mapsto w + it, \end{aligned} \tag{5.98}$$

where $t \in \mathbb{R}$. Let now $j = n$. Transform $\mathcal{C}'$ into $\mathcal{C}$ by means of the map

$$z \mapsto \frac{z-1}{z+1},$$

$$w \mapsto \left(\frac{2}{z+1}\right)^{2/n} w.$$

Then $\mathcal{W}_n$ is the subgroup of $\mathfrak{V}_n$ that transforms into the group of maps

$$\begin{aligned} z &\mapsto e^{it} z, \\ w &\mapsto e^{it/n} w, \end{aligned} \tag{5.99}$$

where $0 \le t < 2\pi n$.

Observe that – upon identifying $G(M)$ with $\mathfrak{V}_j$ for a particular value of j – for every $q \in O(p)$ every element of $G(M)$ can be written in the form gh with $g \in \mathcal{W}_j$, $h \in H_q$, and $\mathcal{W}_j \cap H_q = \{\mathrm{id}\}$. Since no element in a sufficiently small neighborhood of the identity in $\mathcal{W}_j$ lies in the normalizer of H_q in $G(M)$, for every $q \in O(p)$ there exists a tubular neighborhood $\mathcal{U}$ of $O(p)_q$ in $O(p)$ with the following property: for every curve $\gamma \subset \mathcal{U}$ transversal to the leaves $O(p)_{q'}$ for $q' \in \mathcal{U}$ and every $q_1, q_2 \in \gamma$, $q_1 \neq q_2$, we have $H_{q_1} \neq H_{q_2}$. Further, for every two points $q_1, q_2 \in O(p)$ there exists $g \in \mathcal{W}_j$ such that $gO(p)_{q_1} = O(p)_{q_2}$. If for some $q \in O(p)$ there is a non-trivial $g \in \mathcal{W}_j$ such that $gO(p)_q = O(p)_q$, then $gH_qg^{-1} = H_q$ and hence g has the form

for $j = \infty$:

$$\begin{aligned} z &\mapsto z, \\ w &\mapsto w + 2\pi i k_0, \quad k_0 \in \mathbb{Z} \setminus \{0\}, \end{aligned}$$

for $j = n \in \mathbb{N}$:

$$\begin{aligned} z &\mapsto z, \\ w &\mapsto e^{2\pi i k_0/n} w, \quad k_0 \in \mathbb{N},\, k_0 \leq n-1. \end{aligned}$$

It then follows that g lies in the centralizer of $H_{q'}$ for every $q' \in O(p)$. Let $h \in H_q$ be such that $hg \in I_q$. Every element of I_q has finite order (see (ii) of Proposition 4.2), which implies that each of h and g is of finite order. At the same time, if $G(M) = \mathfrak{V}_\infty$, then g is clearly of infinite order; hence $gO(p)_q \neq O(p)_q$ for every $q \in O(p)$ and every non-trivial $g \in \mathcal{W}_\infty$. Assume now that $G(M) = \mathfrak{V}_n$ for some $n \in \mathbb{N}$. Since every non-trivial element of $\mathcal{T}$ has infinite order, we obtain $h = \mathrm{id}$ and thus $g \in I_q$. This argument can be applied to any point in $O(p)_q$, and thus we obtain that g fixes every point in $O(p)_q$. Hence $g = g_q$, where, as before, g_q denotes the element of I_q corresponding to the non-trivial element in $\mathbb{Z}_2$ (see (ii) of Proposition 4.2). Then if a point $q_1 \notin O(p)_q$ is sufficiently close to q, the point $q_2 := gq_1$ is also close to q, and we can assume that $q_1, q_2 \in \mathcal{U}$. It follows from the explicit form of the action of the linear isotropy subgroup L_p on $T_p(M)$ that $q_1 \neq q_2$ and that q_1, q_2 lie on a curve transversal to every leaf in $\mathcal{U}$; hence $H_{q_1} \neq H_{q_2}$. At the same time, we have $H_{q_2} = g_q H_{q_1} g_q^{-1} = H_{q_1}$. This contradiction shows that $gO(p)_q \neq O(p)_q$ for every $q \in O(p)$ and every non-trivial $g \in \mathcal{W}_n$.

Suppose that $G(M) = \mathfrak{V}_\infty$. Fix $q_0 \in O(p)$ for which $H_{q_0} = H_0$, where H_0 is the subgroup of $G(M)$ with Lie algebra generated by X_3, Y_3, and let $f : O(p)_{q_0} \to \mathcal{P}$ be a holomorphic equivalence that transforms $H_{q_0}|_{O(p)_{q_0}}$ into the group $\mathcal{T}$ and such that $\hat{X}_3|_{O(p)_{q_0}}$ and $\hat{Y}_3|_{O(p)_{q_0}}$ are transformed into the vector fields $\pm X_1|_{\mathcal{P}}$ and $Y_1|_{\mathcal{P}}$, respectively, where $\hat{X}_3$, $\hat{Y}_3$ are the vector fields on $O(p)$ corresponding to X_3, Y_3. For every $q \in O(p)$ we now find the unique map $g \in \mathcal{W}_\infty$ such that $gO(p)_{q_0} = O(p)_q$ and define $F(q) := \Big(f(g^{-1}(q)), it\Big) \in \mathbb{C}^2$, where t is the parameter value corresponding to g (see (5.98)). Clearly, F is a real-analytic CR-isomorphism between $O(p)$ and $\mathcal{O}_1$ that transforms $\hat{X}_3 - 1/2\, \hat{Z}_3$ into $Z_1|_{\mathcal{O}_1}$, where $\hat{Z}_3$ is the vector field on $O(p)$ corresponding to Z_3 (recall that $\mathcal{W}_\infty = \{\exp\left(s\left(X_3 - 1/2\, Z_3\right)\right),\, s \in \mathbb{R}\}$), and transforms $\hat{X}_3$, $\hat{Y}_3$

on a neighborhood of $O(p)$ into holomorphic vector fields $\tilde{X}$, $\tilde{Y}$ of the forms appearing in the right-hand sides of formulas (5.91), (5.92), respectively, where ρ, σ, μ, τ are holomorphic in a neighborhood of $\mathcal{O}_1$ and satisfy (5.93). Since $[\hat{X}_3 - 1/2\,\hat{Z}_3, \hat{X}_3] = -1/2\,\hat{Y}_3$ and $[\hat{X}_3 - 1/2\,\hat{Z}_3, \hat{Y}_3] = \hat{X}_3 + 1/2\,\hat{Z}_3$ on $O(p)$, on a neighborhood of $\mathcal{O}_1$ we obtain

$$[Z_1, \tilde{X}] = -\frac{1}{2}\tilde{Y},\ [Z_1, \tilde{Y}] = 2\tilde{X} - Z_1. \tag{5.100}$$

Conditions (5.93), (5.100) uniquely determine the functions ρ, σ, μ, τ as follows:

$$\rho = \pm\frac{i}{4}\Big((z+2)e^{\pm w} - (z-2)e^{\mp w}\Big) \mp i, \quad \sigma = -\frac{i}{4}\Big(e^{w} + e^{-w} - 2\Big),$$

$$\mu = \frac{1}{2}\Big((z+2)e^{\pm w} + (z-2)e^{\mp w}\Big) - z, \quad \tau = -\frac{1}{2}\Big(e^{w} - e^{-w}\Big).$$

Thus, we have shown that if M is a (2,3)-manifold such that $G(M)$ is the universal cover of $SO_{2,1}(\mathbb{R})^0$ and $O(p)$ is a Levi-flat $G(M)$-orbit in M, then there exists a CR-isomorphism from $O(p)$ onto $\mathcal{O}_1$ that transforms near $O(p)$ vector fields from the Lie algebra $\mathfrak{g}(M)$ into vector fields near $\mathcal{O}_1$ from the Lie algebra $\mathfrak{a}^{(\infty)}$ generated by Z_1 and

$$i\Big((z+2)e^{w} - (z-2)e^{-w}\Big)\partial/\partial z - i\Big(e^{w} + e^{-w}\Big)\partial/\partial w,$$

$$\Big((z+2)e^{w} + (z-2)e^{-w}\Big)\partial/\partial z - \Big(e^{w} - e^{-w}\Big)\partial/\partial w.$$

The CR-isomorphism is either the map F constructed above or the map $S \circ F$, where S is given by (5.81).

Let $\mathcal{M}$ be any of the (2,3)-manifolds $\mathfrak{D}_s^{(\infty)}$, $\mathfrak{D}_{s,t}^{(\infty)}$ introduced in **(11)(d)**. The group $G(\mathcal{M})$ coincides with $\mathcal{R}^{(\infty)}$ (see (5.58)) and hence is isomorphic to the universal cover of $SO_{2,1}(\mathbb{R})^0$. Furthermore, $\mathcal{O}_0^{(\infty)}$ (see (5.76)) is a Levi-flat $G(\mathcal{M})$-orbit in $\mathcal{M}$. Hence, as we have shown above, there exists a CR-isomorphism from $\mathcal{O}_0^{(\infty)}$ onto $\mathcal{O}_1$ that transforms $\mathfrak{g}(\mathcal{M})$ near $\mathcal{O}_0^{(\infty)}$ into $\mathfrak{a}^{(\infty)}$ near $\mathcal{O}_1$. Therefore, there exists a CR-isomorphism from $O(p)$ onto $\mathcal{O}_0^{(\infty)}$ that transforms $G(M)|_{O(p)}$ into $\mathcal{R}^{(\infty)}|_{\mathcal{O}_0^{(\infty)}}$.

Suppose that $G(M) = \mathfrak{V}_n$ for $n \in \mathbb{N}$. Fix $q_0 \in O(p)$ for which $H_{q_0} = H_0$, where, as before, H_0 is the subgroup of $G(M)$ with Lie algebra generated by X_3, Y_3, and let $f : O(p)_{q_0} \to \mathcal{P}$ be a holomorphic equivalence that transforms $H_{q_0}|_{O(p)_{q_0}}$ into the group $\mathcal{T}$ and such that $\hat{X}_3|_{O(p)_{q_0}}$ and $\hat{Y}_3|_{O(p)_{q_0}}$ are transformed into the vector fields $\pm X_1|_{\mathcal{P}}$ and $Y_1|_{\mathcal{P}}$, respectively. For every $q \in O(p)$ we now find the unique map $g \in \mathcal{W}_n$ such that $gO(p)_{q_0} = O(p)_q$ and define $F(q) := \Big(f(g^{-1}(q)), e^{it/n}\Big) \in \mathbb{C}^2$, where t is the parameter value corresponding to g (see (5.99)). Clearly, F is a real-analytic CR-isomorphism between $O(p)$ and $\mathcal{O}_1'$ that transforms $\hat{X}_3 - 1/2\,\hat{Z}_3$ into $1/n\,Z_1'|_{\mathcal{O}_1'}$ and transforms $\hat{X}_3$, $\hat{Y}_3$ on a neighborhood of $O(p)$ into holomorphic vector fields $\tilde{X}$, $\tilde{Y}$ of the forms appearing in the right-hand sides of formulas (5.91), (5.92), respectively, where ρ, σ, μ, τ are functions holomorphic near $\mathcal{O}_1'$ and satisfying (5.95). Arguing as before, we obtain

$$\left[\frac{1}{n}\,Z_1', \tilde{X}\right] = -\frac{1}{2}\tilde{Y},\ \left[\frac{1}{n}\,Z_1', \tilde{Y}\right] = 2\tilde{X} - \frac{1}{n}\,Z_1'. \tag{5.101}$$

Conditions (5.95), (5.101) uniquely determine the functions ρ, σ, μ, τ as follows:

$$\rho = \pm\frac{i}{4}\left((z+2)w^{\pm n} - (z-2)w^{\mp n}\right) \mp i, \quad \sigma = -\frac{i}{4n}\left(w^{n+1} + w^{1-n} - 2w\right),$$

$$\mu = \frac{1}{2}\left((z+2)w^{\pm n} + (z-2)w^{\mp n}\right) - z, \quad \tau = -\frac{1}{2n}\left(w^{n+1} - w^{1-n}\right).$$

Thus, we have shown that if M is a (2,3)-manifold such that $G(M)$ is an n-sheeted cover of $SO_{2,1}(\mathbb{R})^0$ and $O(p)$ is a Levi-flat $G(M)$-orbit in M, then there exists a CR-isomorphism from $O(p)$ onto $\mathcal{O}_1'$ that transforms near $O(p)$ vector fields from the Lie algebra $\mathfrak{g}(M)$ into vector fields near $\mathcal{O}_1'$ from the Lie algebra $\mathfrak{a}^{(n)}$ generated by Z_1' and

$$i\left((z+2)w^n - (z-2)w^{-n}\right)\partial/\partial z - \frac{i}{n}\left(w^{n+1} + w^{1-n}\right)\partial/\partial w,$$

$$\left((z+2)w^n + (z-2)w^{-n}\right)\partial/\partial z - \frac{1}{n}\left(w^{n+1} - w^{1-n}\right)\partial/\partial w.$$

The CR-isomorphism is either the map F constructed above or the map $S' \circ F$, where S' is given by

$$\begin{aligned} z &\mapsto z, \\ w &\mapsto 1/w. \end{aligned}$$

Let $\mathcal{M}$ be any of the (2,3)-manifolds $\mathfrak{D}_s^{(n)}$, $\mathfrak{D}_{s,t}^{(n)}$, $\hat{\mathfrak{D}}_t^{(1)}$ (here $n = 1$) introduced in **(11)(d)**. The group $G(\mathcal{M})$ coincides with $\mathcal{R}^{(n)}$ (see **(10)**, (5.56)) and hence is an n-sheeted cover of $SO_{2,1}(\mathbb{R})^0$. Furthermore, $\mathcal{O}_0^{(n)}$ (see (5.71), (5.75)) is a Levi-flat $G(\mathcal{M})$-orbit in $\mathcal{M}$. Hence, as we have shown above, there exists a CR-isomorphism from $\mathcal{O}_0^{(n)}$ onto $\mathcal{O}_1'$ that transforms $\mathfrak{g}(\mathcal{M})$ near $\mathcal{O}_0^{(n)}$ into $\mathfrak{a}^{(n)}$ near $\mathcal{O}_1'$. Therefore, there exists a CR-isomorphism from $O(p)$ onto $\mathcal{O}_0^{(n)}$ that transforms $G(M)|_{O(p)}$ into $\mathcal{R}^{(n)}|_{\mathcal{O}_0^{(n)}}$.

The proof of Proposition 5.4 is complete. ∎

Remark 5.5. It is in fact possible to write down a suitable CR-equivalence between $\mathcal{O}_0^{(j)}$ for $j \in \{1, 2, \ldots, \infty\}$ and either $\mathcal{O}_1$ or $\mathcal{O}_1'$ explicitly. For example, let us realize $\mathcal{O}_0^{(1)}$ as $\Delta \times \partial\Delta \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ (see **(11)(c)**). Then the map given by

$$z = -2\frac{ZW+1}{ZW-1},$$

$$w = W$$

takes $\Delta \times \partial\Delta$ onto $\mathcal{O}_1'$ and transforms near $\Delta \times \partial\Delta$ the Lie algebra of $SU_{1,1}/\{\pm\mathrm{id}\}$-vector fields on $\mathbb{CP}^1 \times \mathbb{CP}^1$ into $\mathfrak{a}^{(1)}$ near $\mathcal{O}_1'$ (here we set $Z_0 = W_0 = 1$ on $\Delta \times \partial\Delta$ and denote $Z := Z_1$, $W := W_1$).

We will now prove the following theorem that finalizes our classification of (2,3)-manifolds in the case when every orbit has codimension 1. In the formulation below we use the notation introduced in Sect. 5.1.

Theorem 5.6. ([I5])*Let M be a (2,3)-manifold. Assume that the $G(M)$-orbit of every point in M is of codimension 1 and that at least one orbit is Levi-flat. Then M is holomorphically equivalent to one of the following manifolds:*

(i) $R_{\alpha,s,t}$, *where* $\alpha \in \mathbb{R}$, $\alpha \neq 0, 1$, *and either* $s = -\infty$, $t = 1$ *or* $s = -1$, $0 < t \leq \infty$, *and in the latter case* $t \neq 1$, *if* $\alpha = 1/2$;
(ii) $\hat{R}_{\alpha,-1,t}$, *where* $\alpha > 0$, $\alpha \neq 1$, $0 < t < \infty$;
(iii) $\hat{U}_{1,t}$, *where* $-\infty < t < 0$;
(iv) $\mathfrak{D}_{s,t}^{(j)}$, *where* $j \in \{1, 2, \dots, \infty\}$, $-1 \leq s < 1 < t \leq \infty$, *and* $s = -1$ *and* $t = \infty$ *do not hold simultaneously.*

The manifolds on list (i)–(iv) are pairwise holomorphically non-equivalent.

Proof: The proof is based on Proposition 5.4 and the orbit gluing procedure introduced in Sect. 3.4.

Observe that the set $\mathfrak{L} := \{p \in M : O(p) \text{ is Levi-flat}\}$ is closed in M. Hence, if $\mathfrak{L}$ is also open, then every orbit in M is Levi-flat. Let $p \in \mathfrak{L}$ and suppose first that there exists a CR-isomorphism $f : O(p) \to \mathcal{O}_1$ that transforms $G(M)|_{O(p)}$ into the group $G_0|_{\mathcal{O}_1}$. The group G_0 acts on $\mathcal{C}' = \left\{(z,w) \in \mathbb{C}^2 : \mathrm{Re}\, z > 0\right\}$; every orbit of this action has the form

$$b_r' := \{(z,w) \in \mathcal{C}' : \mathrm{Re}\, w = r\},$$

for $r \in \mathbb{R}$, and hence is Levi-flat. Arguing as at step (II) of the orbit gluing procedure, we extend f to a biholomorphic map between a $G(M)$-invariant neighborhood U of $O(p)$ and a G_0-invariant neighborhood of $\mathcal{O}_1$ in $\mathcal{C}'$ that satisfies (3.16) for all $g \in G(M)$ and $q \in U$, where $\varphi : G(M) \to G_0$ is an isomorphism. Since every G_0-orbit in $\mathcal{C}'$ is Levi-flat, the set $\mathfrak{L}$ is open. The group G_0 is not isomorphic to any of the groups G_α for $\alpha \in \mathbb{R}^*$, G_0', $\mathfrak{G}$, $\mathcal{R}^{(j)}$, and it follows that every orbit $O(q)$ in M is CR-equivalent to $\mathcal{O}_1$ by means of a map that transforms $G(M)|_{O(q)}$ into $G_0|_{\mathcal{O}_1}$.

We will now further utilize the orbit gluing procedure from Sect. 3.4. Our aim is to show that M is holomorphically equivalent to a G_0-invariant domain in $\mathcal{C}'$. First of all, we need to prove that the map F arising at step (III) extends to a holomorphic automorphism of $\mathcal{C}'$. This map establishes a CR-isomorphism between b'_{r_1} and b'_{r_2} for some $r_1, r_2 \in \mathbb{R}$. Clearly, F has the form $F = \nu \circ g$, where ν is a real translation in w, and $g \in \mathrm{Aut}_{CR}(b'_{r_1})$. Since $F = f_1 \circ f^{-1}$ and the maps f and f_1 transform the group $G(M)|_{O(s)}$ for some $s \in M$ into the groups $G_0|_{b'_{r_1}}$ and $G_0|_{b'_{r_2}}$, respectively, the element g lies in the normalizer of $G_0|_{b'_{r_1}}$ in $\mathrm{Aut}_{CR}(b'_{r_1})$.

The general form of an element of $\mathrm{Aut}_{CR}(b'_{r_1})$ is

$$(z, r_1 + iv) \mapsto (a_v(z), r_1 + i\mu(v)), \tag{5.102}$$

where $v \in \mathbb{R}$, $a_v \in \mathrm{Aut}(\mathcal{P})$ for every v, and μ is a diffeomorphism of $\mathbb{R}$. We consider g in this form and argue as in Sect. 3.4. We then obtain that a_v for every $v \in \mathbb{R}$ lies in the normalizer of $\mathcal{T}$ in $\mathrm{Aut}(\mathcal{P})$ (see (5.83)), and hence $a_v \in \mathcal{T}$ for all v. Moreover, we obtain: $a_{v_1} a a_{v_1}^{-1} = a_{v_2} a a_{v_2}^{-1}$ for all $a \in \mathcal{T}$ and all $v_1, v_2 \in \mathbb{R}$. Therefore, $a_{v_1}^{-1} a_{v_2}$ lies in the center of $\mathcal{T}$, which is trivial. Hence we obtain that $a_{v_1} = a_{v_2}$ for all v_1, v_2. In addition, there exists $d \in \mathbb{R}^*$ such that $\mu^{-1}(v) + \gamma \equiv \mu^{-1}(v + d\gamma)$, for all $\gamma \in \mathbb{R}$. As in Sect. 3.4, differentiating this identity with respect to γ at 0 gives

$$\mu^{-1}(v) = v/d + t \tag{5.103}$$

for some $t \in \mathbb{R}$. Therefore, F extends to a holomorphic automorphism of $\mathcal{C}'$ as the following map:

$$\begin{aligned} z &\mapsto \lambda z + i\beta, \\ w &\mapsto d(w - it) + \sigma, \end{aligned} \tag{5.104}$$

where $\lambda > 0$, $\beta, \sigma \in \mathbb{R}$ (cf. (3.24)).

Any G_0-invariant domain in $\mathcal{C}'$ is given by

$$\left\{(z, w) \in \mathbb{C}^2 : \mathrm{Re}\, z > 0,\ s < \mathrm{Re}\, w < t\right\},$$

for some $-\infty \le s < t \le \infty$. At step (IV) we observe that, since $\mathcal{O}_1$ splits $\mathcal{C}'$, for V sufficiently small we have $V = V_1 \cup V_2 \cup O(x)$, where V_j are open connected non-intersecting sets. If $V_j \subset D$ for $j = 1, 2$, then for the domain D we have $s > -\infty$, $t < \infty$, and the argument applied above to the map F shows that $\hat{F}$ has the form (5.104). Further, using the fact that $\hat{F}$ is G_0-equivariant, we obtain that $\hat{F}$ is a translation in w and that $\mathcal{C}$ covers M, contradicting with the hyperbolicity of M. It then follows that M is equivalent to Δ^2 which is impossible, since $d(\Delta^2) = 6$.

Next, if for $p \in \mathfrak{L}$ there exists a CR-isomorphism between $O(p)$ and $\mathcal{O}'_1$ that transforms $G(M)|_{O(p)}$ into the group $G'_0|\mathcal{O}'_1$, a similar argument gives that M is holomorphically equivalent to the product of $\Delta \times A$, where A is either an annulus or a punctured disk. This is impossible either since $d(\Delta \times A) = 4$.

Let now $p \in \mathfrak{L}$ and suppose that there exists a CR-isomorphism $f : O(p) \to \mathcal{O}_1$ that transforms $G(M)|_{O(p)}$ into the group $G_1|_{\mathcal{O}_1}$. The group G_1 acts on

$$\mathcal{G} := \mathbb{C}^2 \setminus \left\{(z,w) \in \mathbb{C}^2 : \operatorname{Re} z = \operatorname{Re} w = 0\right\},$$

with codimension 1 orbits, and, as as before, we can extend f to a biholomorphic map between a $G(M)$-invariant neighborhood U of $O(p)$ and a G_1-invariant neighborhood of $\mathcal{O}_1$ in $\mathcal{G}$ that satisfies (3.16) for all $g \in G(M)$ and $q \in U$, where $\varphi : G(M) \to G_1$ is an isomorphism. A G_1-orbit in $\mathcal{G}$ is either of the form

$$\left\{(z,w) \in \mathbb{C}^2 : \operatorname{Re} w = r\operatorname{Re} z,\ \operatorname{Re} z > 0\right\},$$

or of the form

$$\left\{(z,w) \in \mathbb{C}^2 : \operatorname{Re} w = r\operatorname{Re} z,\ \operatorname{Re} z < 0\right\},$$

for $r \in \mathbb{R}$, or coincides with either $\hat{\mathcal{O}}_1$ (see (5.5)), or

$$\hat{\mathcal{O}}_1^- := \left\{(z,w) \in \mathbb{C}^2 : \operatorname{Re} z = 0,\ \operatorname{Re} w < 0\right\}, \tag{5.105}$$

and hence is Levi-flat. Therefore, every orbit in M is Levi-flat, and it follows, as before, that every orbit $O(q)$ in M is CR-equivalent to $\mathcal{O}_1$ by means of a map that transforms $G(M)|_{O(q)}$ into $G_1|_{\mathcal{O}_1}$.

In order to show that M is holomorphically equivalent to a G_1-invariant domain in $\mathcal{G}$, we need to deal with steps (III) and (IV) of the orbit gluing procedure. In this case we have $F = \nu \circ g$, where ν is a map of the form (5.33) with $A \in GL_2(\mathbb{R})$, and $g \in \operatorname{Aut}_{CR}(o)$ for some G_1-orbit o. As before, g lies in the normalizer of $G_1|_o$ in $\operatorname{Aut}_{CR}(o)$. Let $\mathfrak{X}$ be a map of the form (5.33) with $A \in GL_2(\mathbb{R})$ that transforms o into $\mathcal{O}_1$, and $g_{\mathfrak{X}} := \mathfrak{X} \circ g \circ \mathfrak{X}^{-1}$. Considering $g_{\mathfrak{X}}$ in the general form (5.102) with $r_1 = 0$ we see that for every $\lambda > 0$, $\beta, \gamma \in \mathbb{R}$, the composition $a_{\lambda v+\gamma} \circ a^{\lambda,\beta} \circ a_v^{-1}$, where $a^{\lambda,\beta}(z) := \lambda z + i\beta$, belongs to $\mathcal{T}$ and is independent of v. This implies that $a_v(z) = \lambda_0 z + i(C_1 v + C_2)$ for some $\lambda_0 > 0$, $C_1, C_2 \in \mathbb{R}$. Also, for every $\lambda > 0$, $\gamma \in \mathbb{R}$ there exist $\lambda_1 > 0$, $\gamma_1 \in \mathbb{R}$ such that $\mu\left(\lambda\mu^{-1}(v) + \gamma\right) = \lambda_1 v + \gamma_1$. It then follows, in particular, that either there exist $c \in \mathbb{R}^*$, $d \in \mathbb{R}$ such that $\mu^{-1}(v) + \gamma \equiv \mu^{-1}\left(e^{c\gamma} v + d(1 - e^{c\gamma})\right)$, or there exists $d \in \mathbb{R}^*$ such that $\mu^{-1}(v) + \gamma \equiv \mu^{-1}(v + d\gamma)$ for all $\gamma \in \mathbb{R}$. Differentiating these identities with respect to γ at 0, we see that the first identity cannot hold and that μ^{-1}, as before, has the form (5.103) for some $t \in \mathbb{R}$. It then follows that $g_{\mathfrak{X}}$ extends to a holomorphic automorphism of $\mathcal{G}$ as the map

$$\begin{aligned} z &\mapsto \lambda_0 z + C_1 w + iC_2, \\ w &\mapsto dw - idt, \end{aligned} \tag{5.106}$$

and thus F extends to an automorphism of $\mathcal{G}$ as well.

Any hyperbolic G_1-invariant domain in $\mathcal{G}$ has the form $S + i\mathbb{R}^2$, where S is an angle of size less than π with vertex at the origin in the $(\operatorname{Re} z, \operatorname{Re} w)$-plane. If at step (IV) we have $V_j \subset D$ for $j = 1, 2$, then the argument applied above to the map F shows that $\hat{F}$ has the form (5.10) with $A \in GL_2(\mathbb{R})$. Further, using the fact that $\hat{F}$ is G_1-equivariant, we obtain that $\hat{F} = \mathrm{id}$, which is impossible. This shows that M is holomorphically equivalent to a hyperbolic G_1-invariant domain in $\mathcal{G}$. By means of a suitable linear transformation every such domain is equivalent to the tube domain whose base is the first quadrant, and thus M is holomorphically equivalent to Δ^2, which is impossible.

Let $p \in \mathfrak{L}$ and suppose that there exists a CR-isomorphism $f : O(p) \to \mathcal{O}_1$ that transforms $G(M)|_{O(p)}$ into the group $G_\alpha|_{\mathcal{O}_1}$ for some $\alpha \in \mathbb{R}^*$, $\alpha \neq 1$. The group G_α acts on $\mathcal{G}$, and every G_α-orbit in $\mathcal{G}$ is either strongly pseudoconvex and has one of the forms

$$\left\{(z,w) \in \mathbb{C}^2 : \mathrm{Re}\, w = r\,(\mathrm{Re}\, z)^\alpha\,,\ \mathrm{Re}\, z > 0\right\},$$

$$\left\{(z,w) \in \mathbb{C}^2 : \mathrm{Re}\, w = r\,(-\mathrm{Re}\, z)^\alpha\,,\ \mathrm{Re}\, z < 0\right\},$$

for $r \in \mathbb{R}^*$, or coincides with one of $\mathcal{O}_1$, $\hat{\mathcal{O}}_1$ (see (5.5)), $\hat{\mathcal{O}}_1^-$ (see (5.105)), and

$$\mathcal{O}_1^- := \left\{(z,w) \in \mathbb{C}^2 : \mathrm{Re}\, z < 0,\ \mathrm{Re}\, w = 0\right\}. \tag{5.107}$$

It then follows that every Levi-flat orbit in M has a $G(M)$-invariant neighborhood in which every other orbit is strongly pseudoconvex. Among the groups G_β (with $\beta \in \mathbb{R}^*$, $\beta \neq 1$, $\beta \neq \alpha$), $\mathfrak{G}$, $\mathcal{R}^{(j)}$ the only group isomorphic to G_α is $G_{1/\alpha}$. Thus, it follows that every Levi-flat orbit $O(q)$ in M is CR-equivalent to $\mathcal{O}_1$ by means of a map that transforms $G(M)|_{O(q)}$ into either $G_\alpha|_{\mathcal{O}_1}$ or $G_{1/\alpha}|_{\mathcal{O}_1}$. In the latter case interchanging the variables we obtain a map that takes $O(q)$ into $\hat{\mathcal{O}}_1$ and transforms $G(M)|_{O(q)}$ into $G_\alpha|_{\hat{\mathcal{O}}_1}$. Next, by Proposition 5.3 , every strongly pseudoconvex orbit $O(q')$ is CR-equivalent to τ_α (see (5.3)) by means of a CR-map that transforms $G(M)|_{O(q')}$ into $G_\alpha|_{\tau_\alpha}$.

We now turn to step (III) of the orbit gluing procedure. For the point $x \in \partial D$ there exists a real-analytic CR-isomorphism f_1 between $O(x)$ and one of $\mathcal{O}_1$, $\hat{\mathcal{O}}_1$, τ_α that transforms $G(M)|_{O(x)}$ into one of $G_\alpha|_{\mathcal{O}_1}$, $G_\alpha|_{\hat{\mathcal{O}}_1}$, $G_\alpha|\tau_\alpha$, respectively. In each of these three cases the corresponding point s can be chosen so that $O(s)$ is strongly pseudoconvex. Then F is a CR-isomorphism between strongly pseudoconvex G_α-orbits, and thus has the form $F = \nu \circ g$, where ν is a map of the form

$$\begin{aligned} z &\mapsto \pm z,\\ w &\mapsto dw \end{aligned} \tag{5.108}$$

with $d \in \mathbb{R}^*$, and $g \in G_\alpha$. Therefore, F extends to an automorphism of $\mathcal{G}$.

Suppose that at step (IV) we have $V_j \subset D$ for $j = 1, 2$. Assume first $O(x)$ is strongly pseudoconvex. Then $\hat{F} = \nu \circ g$, where ν is a non-trivial map of the form (5.108), and $g \in G_\alpha$. Now using the fact that $\hat{F}$ is G_α-equivariant, we obtain that $\hat{F} = \mathrm{id}$, which is impossible. Suppose now that $O(x)$ is Levi-flat. Then $\hat{F} = \nu \circ g$, where ν is one of the maps

$$\begin{array}{llll} z \mapsto -z, & z \mapsto z, & z \mapsto \pm w, & z \mapsto w,\\ w \mapsto w, & w \mapsto -w, & w \mapsto z, & w \mapsto \pm z, \end{array} \tag{5.109}$$

and $g \in \mathrm{Aut}_{CR}(o)$, where o is a Levi-flat G_α-orbit. In this case, either g lies in the normalizer of $G_\alpha|_o$ in $\mathrm{Aut}_{CR}(o)$, or $gG_\alpha|_o g^{-1} = G_{1/\alpha}|_o$ and $gG_{1/\alpha}|_o g^{-1} = G_\alpha|_o$. Transforming o into $\mathcal{O}_1$ by a map $\mathfrak{X}$ from list (5.109) and arguing as in the case of the group G_1 for the map F, we obtain that $g_{\mathfrak{X}} := \mathfrak{X} \circ g \circ \mathfrak{X}^{-1}$ extends to a holomorphic automorphism of $\mathcal{G}$ as a map of the form (5.106) with $C_1 = 0$. It then follows that $\hat{F}$ has the form (5.10) with $A \in GL_2(\mathbb{R})$. Now, using the G_α-equivariance of $\hat{F}$ we again see that $\hat{F} = \mathrm{id}$ which is impossible. Hence M is holomorphically equivalent to a G_α-invariant domain in $\mathcal{G}$, and we obtain (i) and (ii) of the theorem.

Let $p \in \mathfrak{L}$ and suppose that there exists a CR-isomorphism $f : O(p) \to \mathcal{O}_1$ that transforms $G(M)_{O(p)}$ into the group $\mathfrak{G}|_{\mathcal{O}_1}$. The group $\mathfrak{G}$ acts on $\mathcal{G}$, and every $\mathfrak{G}$-orbit in $\mathcal{G}$ is either strongly pseudoconvex and has one of the forms

$$\left\{(z,w) \in \mathbb{C}^2 : \mathrm{Re}\, z = \mathrm{Re}\, w \ln\left(r \mathrm{Re}\, w\right),\ \mathrm{Re}\, w > 0\right\},$$

$$\left\{(z,w) \in \mathbb{C}^2 : \mathrm{Re}\, z = \mathrm{Re}\, w \ln\left(-r \mathrm{Re}\, w\right),\ \mathrm{Re}\, w < 0\right\},$$

for $r > 0$, or coincides with one of $\mathcal{O}_1$, $\mathcal{O}_1^-$ (see (5.107)).

It then follows that every Levi-flat orbit in M has a $G(M)$-invariant neighborhood in which every other orbit is strongly pseudoconvex, that every Levi-flat orbit in M is CR-equivalent to $\mathcal{O}_1$ by means of a map that transforms $G(M)|_{O(p)}$ into $\mathfrak{G}|_{\mathcal{O}_1}$ and that every strongly pseudoconvex orbit is CR-equivalent to ξ (see (5.8)) by means of a map that transforms $G(M)|_{O(p)}$ into $\mathfrak{G}|_{\xi}$.

At step (III), as in the case of the groups G_α above, we can choose s so that $O(s)$ is strongly pseudoconvex. It then follows that $F = \nu \circ g$, where ν is a map of the form

$$\begin{aligned} z &\mapsto dz, \\ w &\mapsto dw \end{aligned} \tag{5.110}$$

with $d \in \mathbb{R}^*$, and $g \in \mathfrak{G}$. Therefore, F extends to an automorphism of $\mathcal{G}$.

Suppose that at step (IV) we have $V_j \subset D$ for $j = 1, 2$. Assume first $O(x)$ is strongly pseudoconvex. Then $\hat{F} = \nu \circ g$, where ν is a non-trivial map of the form (5.110), and $g \in \mathfrak{G}$. Now, using the fact that $\hat{F}$ is $\mathfrak{G}$-equivariant, we obtain that $\hat{F} = \mathrm{id}$, which is impossible. Suppose now that $O(x)$ is Levi-flat. Then $\hat{F} = \nu \circ g$, where ν is map (5.110) with $d = -1$, and $g \in \mathrm{Aut}_{CR}(o)$, where o is a Levi-flat $\mathfrak{G}$-orbit. The element g lies in the normalizer of $\mathfrak{G}$ in $\mathrm{Aut}_{CR}(o)$. Transforming o into $\mathcal{O}_1$ by a map $\mathfrak{X}$ of the form (5.110) with $d = \pm 1$ and considering $g_{\mathfrak{X}} := \mathfrak{X} \circ g \circ \mathfrak{X}^{-1}$ in the general form (5.102) with $r_1 = 0$, we obtain, as before, that μ^{-1} has the form (5.103) for some $d \in \mathbb{R}^*$, $t \in \mathbb{R}$, and that $a_v(z) = \lambda_0 z + i\beta(v)$, where $\lambda_0 > 0$ and $\beta(v)$ is a function satisfying for every $\lambda > 0$ and $\gamma \in \mathbb{R}$ the following condition:

$$\partial/\partial v \left[\beta\left(\lambda\mu^{-1}(v) + \gamma\right) - \lambda\beta\left(\mu^{-1}(v)\right) + \lambda \ln\lambda \left(\lambda_0 \mu^{-1}(v) - v\right)\right] \equiv 0.$$

Setting $\lambda = 1$ in the above identity gives that $g_{\mathfrak{X}}$ extends to an automorphism of $\mathcal{G}$ as a map of the form (5.106). Therefore, $\hat{F}$ has the form (5.10) with

$A \in GL_2(\mathbb{R})$, and using the $\mathfrak{G}$-equivariance of $\hat{F}$ we again see that $\hat{F} = \mathrm{id}$ which is impossible. Hence M is holomorphically equivalent to a $\mathfrak{G}$-invariant domain in $\mathcal{G}$, and we have obtained (iii) of the theorem.

Let $p \in \mathfrak{L}$ and suppose that there exists a CR-isomorphism $f : O(p) \to \mathcal{O}_0^{(j)}$ that transforms $G(M)|_{O(p)}$ into the group $\mathcal{R}^{(j)}|_{\mathcal{O}_0^{(j)}}$ for some $j \in \{1, 2, \dots, \infty\}$. The group $\mathcal{R}^{(j)}$ acts on $\mathfrak{D}^{(j)}$, where $\mathfrak{D}^{(1)} := \mathfrak{D}^{(1)}_{-1,\infty}$ (see (5.74)), $\mathfrak{D}^{(j)} := M^{(j)} \setminus \mathcal{O}^{(2j)}$ for $1 < j < \infty$, and $\mathfrak{D}^{(\infty)} := M^{(\infty)}$ (see **(11)(a), (d)**). Apart from $\mathcal{O}_0^{(j)}$, every $\mathcal{R}^{(j)}$-orbit in $\mathfrak{D}^{(j)}$ is strongly pseudoconvex and is one of $\nu_\alpha^{(j)}$, for $-1 < \alpha < 1$, or $\eta_\alpha^{(2j)}$, for $\alpha > 1$. It then follows that Levi-flat orbits in M are isolated, and every such orbit $O(q)$ is CR-equivalent to $\mathcal{O}_0^{(j)}$ by means of a CR-isomorphism that transforms $G(M)|_{O(q)}$ into the group $\mathcal{R}^{(j)}|_{\mathcal{O}_0^{(j)}}$.

At step (III) we again choose s so that $O(s)$ is strongly pseudoconvex which gives that F extends to $\mathfrak{D}^{(j)}$ as an element of $\mathcal{R}^{(j)}$. At step (IV), we observe that $O(x)$ cannot be strongly pseudoconvex since otherwise $\hat{F}$ would be a CR-isomorphism between two distinct strongly pseudoconvex $\mathcal{R}^{(j)}$-orbits in $\mathfrak{D}^{(j)}$, while in fact $\mathcal{R}^{(j)}$-orbits are pairwise CR non-equivalent. On the other hand, $O(x)$ cannot be Levi-flat either, since otherwise $\hat{F}$ would be a CR-isomorphism between two distinct Levi-flat $\mathcal{R}^{(j)}$-orbits in $\mathfrak{D}^{(j)}$, while $O_0^{(j)}$ is the only Levi-flat orbit in $\mathfrak{D}^{(j)}$. This implies that M is holomorphically equivalent to a $\mathcal{R}^{(j)}$-invariant domain in $\mathfrak{D}^{(j)}$ which leads to (iv) of the theorem.

The proof of Theorem 5.6 is complete. ■

5.4 Codimension 2 Orbits

In this section we finalize our classification by allowing codimension 2 orbits to be present in the manifold. We will prove the following theorem (as before, we use the notation introduced in Sect. 5.1).

Theorem 5.7. ([I5])*Let M be a (2,3)-manifold. Assume that a $G(M)$-orbit of codimension 1 and a $G(M)$-orbit of codimension 2 are present in M. Then M is holomorphically equivalent to one of the following manifolds:*

(i) $\mathfrak{S}_1$;
(ii) E_t *with* $1 < t < \infty$;
(iii) $E_t^{(2)}$ *with* $1 < t < \infty$;
(iv) Ω_t *with* $-1 < t < 1$;
(v) D_s *with* $1 \le s < \infty$;
(vi) $D_s^{(2)}$ *with* $1 < s < \infty$;
(vii) $D_s^{(n)}$ *with* $n \ge 3$, $1 \le s < \infty$;
(viii) $\mathfrak{D}_s^{(n)}$ *with* $n \ge 1$, $-1 < s < 1$;
(ix) $\hat{\mathfrak{D}}_t^{(1)}$ *with* $1 < t < \infty$.

Proof: Since a codimension 1 orbit is present in M, it follows that there are at most two codimension 2 orbits (see [AA]). Let O be one such orbit. Parts (iii) and (iv) of Proposition 4.2 yield that for every $p \in O$ the group I_p^0 is isomorphic to U_1 (in particular, $G(M)$ has a subgroup isomorphic to U_1), and there exists an I_p^0-invariant connected complex curve C_p in M that intersects O transversally at p. If O is a complex curve, one such curve C_p corresponds – upon local linearization of the I_p-action – to the L_p^0-invariant subspace $\{w = 0\}$ of $T_p(M)$, where the coordinates (z, w) in $T_p(M)$ are chosen with respect to the decomposition of $T_p(M)$ specified in (iii) of Proposition 4.2, with $\{z = 0\}$ corresponding to $V = T_p(O(p))$; if, in addition, the isotropy linearization is given by (4.2), then the maximal extension of this curve is the only maximally extended complex curve in M with these properties. If O is a totally real orbit, C_p can be constructed from any of the two L_p^0-invariant subspaces $\{z = \pm iw\}$ of $T_p(M)$, where the coordinates (z, w) in $T_p(M)$ are chosen so that $V = \{\operatorname{Im} z = 0, \operatorname{Im} w = 0\}$ (see (iv) of Proposition 4.2); locally near p there are no other such curves. Clearly, there exists a neighborhood U of p such that $U \cap (C_p \setminus \{p\})$ is equivalent to a punctured disk.

Since there is a codimension 1 orbit in M, the group $G(M)$ is either isomorphic to one of the groups listed in Proposition 5.3 (if a strongly pseudoconvex orbit is present in M) or to one of G_1, G_0, G_0' (if all codimension 1 orbits are Levi-flat – see (5.2) and (5.88)). Since G_0 and G_1 do not contain subgroups isomorphic to U_1, the group $G(M)$ in fact cannot be isomorphic to either of these groups. Let M' be the manifold obtained from M by removing all codimension 2 orbits, and suppose that $G(M)$ is isomorphic to G_0'. The subgroup of G_0' isomorphic to U_1 is unique and consists of all rotations in w, it is normal and maximal compact in G_0'; we denote it by J. It follows from the proof of Theorem 5.6 that M' is holomorphically equivalent to

$$\mathcal{V}_{s,t} := \left\{(z, w) \in \mathbb{C}^2 : \operatorname{Re} z > 0,\ s < |w| < t\right\},$$

where $0 \le s < t \le \infty$, and either $s > 0$ or $t < \infty$, by means of a map f that satisfies (3.16) for all $g \in G(M)$, $q \in M'$ and an isomorphism $\varphi : G(M) \to G_0'$. Clearly, $I_p = I := \varphi^{-1}(J)$ for every $p \in O$. In particular, I_p acts trivially on O for every $p \in O$; hence O is a complex curve with isotropy linearization given by (4.2), and there are no totally real orbits in M. The group G_0' acts on $\tilde{\mathcal{C}}' := \mathcal{P} \times \mathbb{CP}^1$ (we set $g(z, \infty) := (\lambda z + i\beta, \infty)$ for every g of the form (5.88)). This action has two complex curve orbits

$$\begin{aligned} \mathcal{O}_7 &:= \mathcal{P} \times \{0\}, \\ \mathcal{O}_8 &:= \mathcal{P} \times \{\infty\}. \end{aligned} \tag{5.111}$$

It is straightforward to observe that every connected J-invariant complex curve in $\mathcal{V}_{s,t}$ extends to a curve of the form

$$N_{z_0} := \{z = z_0\} \cap \mathcal{V}_{s,t},$$

for some $z_0 \in \mathcal{P}$, which is either an annulus (possibly with infinite outer radius) or a punctured disk. Fix $p_0 \in O$, let C_{p_0} be the unique maximally extended

I-invariant complex curve in M that intersects O at p_0 transversally, and let $z_0 \in \mathcal{P}$ be such that $f(C_{p_0} \setminus \{p_0\}) = N_{z_0}$. Since for a sequence $\{p_j\}$ in C_{p_0} converging to p_0 the sequence $\{f(p_j)\}$ approaches either $\{z = z_0, |w| = s\}$ or $\{z = z_0, |w| = t\}$ and $C_{p_0} \setminus \{p_0\}$ is equivalent to a punctured disk near p_0, we have either $s = 0$ or $t = \infty$, respectively.

Assume first that $s = 0$. We extend f to a map from $\hat{M} := M' \cup O$ onto the domain

$$\mathcal{V}_t := \big\{(z,w) \in \mathbb{C}^2 : \operatorname{Re} z > 0,\ |w| < t\big\} = \mathcal{V}_{0,t} \cup \mathcal{O}_7,$$

by setting $f(p_0) := q_0 := (z_0, 0) \in \mathcal{O}_7$, with z_0 constructed as above. The extended map is 1-to-1 and satisfies (3.16) for all $g \in G(M)$, $q \in \hat{M}$. To prove that f is holomorphic on all of $\hat{M}$, it suffices to show that f is continuous on O. Our argument is similar to that in Sect. 3.4. We will prove that every sequence $\{p_j\}$ in $\hat{M}$ converging to p_0 has a subsequence along which the values of f converge to q_0. Let first $\{p_j\}$ be a sequence in O. Clearly, there exists a sequence $\{g_j\}$ in $G(M)$ such that $p_j = g_j p_0$ for all j. Since $G(M)$ acts properly on M, there exists a converging subsequence $\{g_{j_k}\}$ of $\{g_j\}$, and we denote by g_0 its limit. It then follows that $g_0 \in I$ and, since f satisfies (3.16), we obtain that $\{f(p_{j_k})\}$ converges to q_0. Next, if $\{p_j\}$ is a sequence in M', then there exists a sequence $\{g_j\}$ in $G(M)$ such that $g_j p_j \in C_{p_0}$. Clearly, the sequence $\{g_j p_j\}$ converges to p_0 and hence $\{f(g_j p_j)\}$ converges to q_0. Again, the properness of the $G(M)$-action on M yields that there exists a converging subsequence $\{g_{j_k}\}$ of $\{g_j\}$. Let g_0 be its limit; as before, we have $g_0 \in I$. Property (3.16) now implies $f(p_{j_k}) = [\varphi(g_{j_k})]^{-1} f(g_{j_k} p_{j_k})$, and therefore $\{f(p_{j_k})\}$ converges to q_0. Thus, f is holomorphic on $\hat{M}$. Since $t < \infty$, the orbit O is the only codimension 2 orbit, and thus M is holomorphically equivalent to $\mathcal{V}_t$. This is, however, impossible since $d(\mathcal{V}_t) = 6$.

Assume now that $t = \infty$. Arguing as in the case $s = 0$ and mapping O onto $\mathcal{O}_8$, we can extend f to a biholomorphic map between M and the domain in $\tilde{\mathcal{C}}'$ given by

$$\big\{(z,w) \in \mathbb{C}^2 : \operatorname{Re} z > 0,\ |w| > s\big\} \cup \mathcal{O}_8 = \mathcal{V}_{s,\infty} \cup \mathcal{O}_8,$$

which is holomorphically equivalent to $\mathcal{V}_1$. This is again impossible, and we have ruled out the case when $G(M)$ is isomorphic to G_0'.

It then follows that there is always a strongly pseudoconvex orbit in M and hence $G(M)$ is isomorphic to one of the groups listed in Proposition 5.3. Observe that the groups that arise in subcases (g), (h), (j'), (k), (m'), (n') of case (A) as well as in cases (D) and (F) do not have non-trivial compact subgroups; thus these situations do not in fact occur. In addition, arguing as in the proof of Theorem 5.2, we rule out case (B).

We now assume that a complex curve orbit is present in M. Let O be such an orbit. Then (iii) of Proposition 4.2 gives that O is equivalent to $\mathcal{P}$. Furthermore, if for $p \in O$ the group I_p^0 acts on O non-trivially, that is, L_p^0 is given by some $H^2_{k_1,k_2}$ (see (4.1)), then there exists a finite normal subgroup

$H \subset I_p$ such that $G(M)/H$ is isomorphic to $\mathrm{Aut}(\mathcal{P}) \simeq SO_{2,1}(\mathbb{R})^0$; if I_p^0 acts on O trivially (see (4.2)), then there is a 1-dimensional normal compact subgroup $H \subset I_p$ such that $G(M)/H$ is isomorphic to the subgroup $\mathcal{T} \subset \mathrm{Aut}(\mathcal{P})$ (see (5.83)). In particular, every maximal compact subgroup of $G(M)$ is 1-dimensional and therefore is isomorphic to U_1. It then follows that for every $p \in O$ the group I_p^0 is a maximal compact subgroup of $G(M)$ and hence I_p is connected. Observe now that in subcases (l), (l'), (l") of case (A) as well as in case (C) the group $G(M)$ is compact. In case (G) the group $G(M)$ is isomorphic to $U_1 \times \mathbb{R}^2$; thus no quotient of $G(M)$ by a finite subgroup is isomorphic to $SO_{2,1}(\mathbb{R})^0$ and the quotient of $G(M)$ by its maximal compact subgroup is not isomorphic to $\mathcal{T}$. Furthermore, in subcases (j), (j") the group $G(M)$ is isomorphic to $U_1 \ltimes \mathbb{R}^2$. This group has no 1-dimensional compact normal subgroups and no quotient of this group by a finite subgroup is isomorphic to $SO_{2,1}(\mathbb{R})^0$. Therefore, if a complex curve orbit is present in M, we only need to consider subcases (m), (m"), (n), (n") of case (A), and case (E).

We begin with case (E) and assume first that for some point $p \in M$ there exists a CR-isomorphism between $O(p)$ and the hypersurface ε'_α for some $\alpha > 0$ (see (d) in the proof of Theorem 5.2), that transforms $G(M)|_{O(p)}$ into $G_{\varepsilon'_\alpha}|_{\varepsilon'_\alpha}$, where $G_{\varepsilon'_\alpha}$ is the group of all maps

$$\begin{aligned} z &\mapsto \lambda z + i\beta, \\ w &\mapsto e^{i\psi}\lambda^{1/\alpha} w, \end{aligned} \tag{5.112}$$

with $\lambda > 0$, $\psi, \beta \in \mathbb{R}$.

We proceed as in the case of the group G'_0 considered above. In this case Levi-flat orbits are not present in M, and it follows from the proof of Theorem 5.2 that M' is holomorphically equivalent to $\mathfrak{R}_{\alpha,s,t}$ (see (5.86)) for some $0 \le s < t \le \infty$, with either $s > 0$ or $t < \infty$, by means of a map f that satisfies (3.16) for all $g \in G(M)$, $q \in M'$ and an isomorphism $\varphi : G(M) \to G_{\varepsilon'_\alpha}$. The only 1-dimensional compact subgroup of $G_{\varepsilon'_\alpha}$ is the maximal compact normal subgroup $J^{\varepsilon'_\alpha}$ given by the conditions $\lambda = 1$, $\beta = 0$ in (5.112). Clearly, $I_p = I := \varphi^{-1}(J^{\varepsilon'_\alpha})$ for all $p \in O$, which implies, as before, that O is a complex curve with isotropy linearization given by (4.2), and there are no totally real orbits. The group $G_{\varepsilon'_\alpha}$ acts on $\tilde{\mathcal{C}}'$, and, as before, this action has two complex curve orbits $\mathcal{O}_7$ and $\mathcal{O}_8$ (see (5.111)).

Further, every connected $J^{\varepsilon'_\alpha}$-invariant complex curve in $\mathfrak{R}_{\alpha,s,t}$ extends to a curve of the form

$$\{z = z_0\} \cap \mathfrak{R}_{\alpha,s,t} = \Big\{(z,w) \in \mathbb{C}^2 : z = z_0, \\ (\mathrm{Re}\, z_0/t)^{1/\alpha} < |w| < (\mathrm{Re}\, z_0/s)^{1/\alpha}\Big\},$$

for some $z_0 \in \mathcal{P}$, which is either an annulus or a punctured disk. As before, we obtain that either $s = 0$, or $t = \infty$.

If $t = \infty$, we extend f to a biholomorphic map from $\hat{M} = M' \cup O$ onto the domain

$$\mathfrak{E}'_{\alpha,s} := \left\{(z,w) \in \mathbb{C}^2 : \operatorname{Re} z > s|w|^{\alpha}\right\} = \mathfrak{R}_{\alpha,s,\infty} \cup \mathcal{O}_7 \tag{5.113}$$

(cf. (ii) of Theorem 3.1). Since $s > 0$, the orbit O is the only codimension 2 orbit, and hence M is holomorphically equivalent to $\mathfrak{E}'_{\alpha,s}$. Similarly, if $s = 0$, then M is holomorphically equivalent to the domain

$$\left\{(z,w) \in \mathbb{C}^2 : 0 < \operatorname{Re} z < t|w|^{\alpha}\right\} \cup \mathcal{O}_8 = \mathfrak{R}_{\alpha,0,t} \cup \mathcal{O}_8, \tag{5.114}$$

which is equivalent to the domain

$$\mathcal{E}'_{\alpha} := \left\{(z,w) \in \mathbb{C}^2 : \operatorname{Re} z > 0,\ |w| < (\operatorname{Re} z)^{-1/\alpha}\right\}$$

(cf. (iii) of Theorem 3.1). This is, however, impossible since $d(\mathfrak{E}'_{\alpha,s}) \geq 4$ and $d(\mathcal{E}'_{\alpha}) = 4$.

Assume now that in case (E) for some point $p \in M$ there exists a CR-isomorphism f between $O(p)$ and the hypersurface ε'_{α} for some $\alpha \in \mathbb{Q}$, $\alpha > 0$, that transforms $G(M)|_{O(p)}$ into $\mathfrak{V}_{\alpha}|_{\varepsilon'_{\alpha}}$ (see Proposition 5.3 for the definition of $\mathfrak{V}_{\alpha}$). Let $\alpha = k_1/k_2$, for $k_1, k_2 \in \mathbb{N}$, with $(k_1, k_2) = 1$.

As before, f extends to a biholomorphic map between M' and $\mathfrak{R}_{\alpha,s,t}$, where $0 \leq s < t \leq \infty$, and either $s > 0$ or $t < \infty$. The map f satisfies (3.16) for all $g \in G(M)$, $q \in M'$ and an isomorphism $\varphi : G(M) \to \mathfrak{V}_{\alpha}$. The group $\mathfrak{V}_{\alpha}$ acts on $\tilde{\mathcal{C}}'$, and, as before, this action has two complex curve orbits $\mathcal{O}_7$ and $\mathcal{O}_8$. Every 1-dimensional compact subgroup of $\mathfrak{V}_{\alpha}$ is the isotropy subgroup of the points $(z_0, 0) \in \mathcal{O}_7$ and $(z_0, \infty) \in \mathcal{O}_8$ for a uniquely chosen $z_0 \in \mathcal{P}$. For $z_0 \in \mathcal{P}$ denote by $J_{z_0}^{\varepsilon'_{\alpha}}$ the corresponding maximal compact subgroup of $\mathfrak{V}_{\alpha}$.

For every $z_0 \in \mathcal{P}$ there is a family $\mathcal{F}_{z_0}^{\mathfrak{R}}$ of connected closed complex curves in $\mathfrak{R}_{\alpha,s,t}$ invariant under the $J_{z_0}^{\varepsilon'_{\alpha}}$-action, such that every $J_{z_0}^{\varepsilon'_{\alpha}}$-invariant connected complex curve in $\mathfrak{R}_{\alpha,s,t}$ extends to a curve from $\mathcal{F}_{z_0}^{\mathfrak{R}}$. We will now describe $\mathcal{F}_1^{\mathfrak{R}}$ (here $z_0 = 1$); for arbitrary $z_0 \in \mathcal{P}$ we have $\mathcal{F}_{z_0}^{\mathfrak{R}} = g\left(\mathcal{F}_1^{\mathfrak{R}}\right)$, where $g \in \mathfrak{V}_{\alpha}$ is constructed from an element $\tilde{g} \in \operatorname{Aut}(\mathcal{P})$ such that $z_0 = \tilde{g}(1)$. The family $\mathcal{F}_1^{\mathfrak{R}}$ consists of the curves

$$\left\{(z,w) \in \mathbb{C}^2 : (z^2 - 1)^{k_2} = \rho\, w^{k_1}\right\} \cap \mathfrak{R}_{\alpha,s,t},$$

where $\rho \in \mathbb{C}$. Each of these curves is equivalent to either an annulus or a punctured disk. The latter occurs only for $\rho = 0$ if either $s = 0$ or $t = \infty$, and for $\rho \neq 0$ if $t = \infty$. If either $s = 0$ or $t = \infty$, the corresponding curves accumulate to either the point $(1, \infty) \in \mathcal{O}_8$ or the point $(1, 0) \in \mathcal{O}_7$, respectively.

Fix $p_0 \in O$. Since $\varphi(I_{p_0})$ is a 1-dimensional compact subgroup of $\mathfrak{V}_{\alpha}$, there is a unique $z_0 \in \mathcal{P}$ such that $\varphi(I_{p_0}) = J_{z_0}^{\varepsilon'_{\alpha}}$. Consider any connected I_{p_0}-invariant complex curve C_{p_0} in M intersecting O transversally at p_0. Since

$f(C_{p_0} \setminus \{p_0\})$ is $J_{z_0}^{\varepsilon'_\alpha}$-invariant, it extends to a complex curve $C \in \mathcal{F}_{z_0}^{\mathfrak{R}}$. If a sequence $\{p_j\}$ in $C_{p_0} \setminus \{p_0\}$ accumulates to p_0, the sequence $\{f(p_j)\}$ accumulates to one of the two ends of C, and therefore we have either $s = 0$ or $t = \infty$.

Assume first that $t = \infty$. In this case, arguing as earlier, we can extend f to a biholomorphic map between $\hat{M}$ and $\mathfrak{E}'_{\alpha,s}$ by setting $f(p_0) := q_0 := (z_0, 0) \in \mathcal{O}_7$, where p_0 and z_0 are related as specified above. As before, it is straightforward to show that O is the only codimension 2 orbit in M, and hence it follows that M is holomorphically equivalent to $\mathfrak{E}'_{\alpha,s}$. Similarly, it can be proved that for $s = 0$ the manifold M is holomorphically equivalent to $\mathcal{E}'_\alpha$. As before, this is impossible and thus in case (E) no orbit is a complex curve.

We now consider the remaining subcases of case (A). Suppose first that there is an orbit in M whose model is either some ν_α, or some η_α, or some $\eta_\alpha^{(2)}$. It then follows from the poofs of Theorems 5.2 and 5.6 that M' is holomorphically equivalent to one of the following: $\Omega_{s,t}$ with $-1 \le s < t \le 1$ (see (5.37)); $D_{s,t}$ with $1 \le s < t \le \infty$ (see (5.43)); $D_{s,t}^{(2)}$ with $1 \le s < t \le \infty$ (see (5.60)); $\mathfrak{D}_{s,t}^{(1)}$ with $-1 \le s < 1 < t \le \infty$, where $s = -1$ and $t = \infty$ do not hold simultaneously (see (5.73)).

Suppose first that M' is equivalent to $\Omega_{s,t}$, and let f be an equivalence map. The group $\mathcal{R}_\nu$ (see (5.38)) acts on the domain Ω_1 (see (5.41)) with the totally real codimension 2 orbit $\mathcal{O}_5$ (see (5.42)). Every 1-dimensional compact subgroup of $\mathcal{R}_\nu$ is the isotropy subgroup of a unique point in $\mathcal{O}_5$. For $q_0 \in \mathcal{O}_5$ denote by $J_{q_0}^\nu$ its isotropy subgroup under the action of $\mathcal{R}_\nu$. There is a family $\mathcal{F}_{q_0}^\Omega$ of connected closed complex curves in $\Omega_{s,t}$ invariant under the $J_{q_0}^\nu$-action, such that every connected $J_{q_0}^\nu$-invariant complex curve in $\Omega_{s,t}$ extends to a curve from $\mathcal{F}_{q_0}^\Omega$. As before, it is sufficient to describe this family only for a particular choice of q_0. The family $\mathcal{F}_{(0,0)}^\Omega$ consists of the connected components of non-empty sets of the form

$$\begin{aligned} &\big\{(z,w) \in \mathbb{C}^2 : z^2 + w^2 = \rho\big\} \cap \Omega_{s,t}, \\ &\big\{(z,w) \in \mathbb{C}^2 : z = iw\big\} \cap \Omega_{s,t}, \\ &\big\{(z,w) \in \mathbb{C}^2 : z = -iw\big\} \cap \Omega_{s,t}, \end{aligned} \tag{5.115}$$

where $\rho \in \mathbb{C}^*$. Each of the curves from $\mathcal{F}_{(0,0)}^\Omega$ is equivalent to either an annulus or a punctured disk. The latter is possible only for the last two curves and only for $s = -1$, in which case they accumulate to $(0,0) \in \mathcal{O}_5$.

Now, arguing as in the second part of case (E) above, we obtain that $s = -1$ and extend f to a map from $\hat{M}$ onto Ω_t such that $f(O) = \mathcal{O}_5 \subset \Omega_t$. It can be shown, as before, that f is holomorphic on $\hat{M}$. However, O is a complex curve in $\hat{M}$ whereas $\mathcal{O}_5$ is totally real in Ω_t. Hence M' cannot be equivalent to $\Omega_{s,t}$.

Assume next that M' is equivalent to $D_{s,t}$ by means of a map f. The group R_η (see (5.45)) acts on the domain D_1 (see (5.47)) with the complex curve orbit $\mathcal{O}$ (see (5.44)). We again argue as in the second part of case (E) above. Every 1-dimensional compact subgroup of R_η is the isotropy subgroup of a unique point in $\mathcal{O}$. For $q_0 \in \mathcal{O}$ denote by $J^\eta_{q_0}$ its isotropy subgroup under the action of R_η. There is a family $\mathcal{F}^D_{q_0}$ of connected closed complex curves in $D_{s,t}$ invariant under the $J^\eta_{q_0}$-action, such that every connected $J^\eta_{q_0}$-invariant complex curve in $D_{s,t}$ extends to a curve from $\mathcal{F}^D_{q_0}$. The family $\mathcal{F}^D_{(i,0)}$ consists of the sets

$$\begin{aligned}&\big\{(z,w)\in\mathbb{C}^2 : 1+z^2+\rho w^2=0\big\}\cap D_{s,t},\\&\{w=0\}\cap D_{s,t},\end{aligned}$$

where $\rho \in \mathbb{C}$. Each of the curves from $\mathcal{F}^D_{(i,0)}$ is equivalent to an annulus for $t<\infty$ and to a punctured disk if $t=\infty$, in which case it accumulates to $(i,0)\in\mathcal{O}$.

As before, we now obtain that $t=\infty$ and extend f to a biholomorphic map from $\hat{M}$ onto D_s such that $f(O)=\mathcal{O}$. It is straightforward to see that O is the only codimension 2 orbit; hence M is holomorphically equivalent to D_s, and we have obtained (v) of the theorem.

Suppose now that M' is equivalent to $\mathfrak{D}^{(1)}_{s,t}$, and let f be an equivalence map. The group $\mathcal{R}^{(1)}$ (see **(10)**) acts on $M^{(1)}$ (see (5.70)) with the complex curve orbit $\mathcal{O}^{(2)}$ (see (5.67)) and the totally real orbit $\mathcal{O}_6$ (see (5.72)). Every 1-dimensional compact subgroup of $\mathcal{R}^{(1)}$ is the isotropy subgroup of a unique point in each of $\mathcal{O}^{(2)}$, $\mathcal{O}_6$. For $q_1\in\mathcal{O}^{(2)}$ and $q_2\in\mathcal{O}_6$ that have the same isotropy subgroup under the $\mathcal{R}^{(1)}$-action, denote this subgroup by $J^{\mathfrak{D}}_{q_1,q_2}$. As before, there is a family $\mathcal{F}^{\mathfrak{D}}_{q_1,q_2}$ of connected complex closed curves in $\mathfrak{D}^{(1)}_{s,t}$ invariant under the $J^{\mathfrak{D}}_{q_1,q_2}$-action, such that every connected $J^{\mathfrak{D}}_{q_1,q_2}$-invariant complex curve in $\mathfrak{D}^{(1)}_{s,t}$ extends to a curve from $\mathcal{F}^{\mathfrak{D}}_{q_1,q_2}$. The family $\mathcal{F}^{\mathfrak{D}}_{(0:1:i:0),(0,0,-i)}$ consists of the connected components of the sets

$$\begin{aligned}&\big\{(z_1,z_2,z_3)\in\mathbb{C}^3 : z_1^2+z_2^2+\rho z_3^2=0\big\}\cap\mathfrak{D}^{(1)}_{s,t},\\&\{z_3=0\}\cap\mathfrak{D}^{(1)}_{s,t},\\&\big\{(z_1,z_2,z_3)\in\mathbb{C}^3 : z_1=iz_2\big\}\cap\mathfrak{D}^{(1)}_{s,t},\\&\big\{(z_1,z_2,z_3)\in\mathbb{C}^3 : z_1=-iz_2\big\}\cap\mathfrak{D}^{(1)}_{s,t},\end{aligned}\tag{5.116}$$

where $\rho\in\mathbb{C}^*$. Each of the sets from $\mathcal{F}^{\mathfrak{D}}_{(0:1:i:0),(0,0,-i)}$ is equivalent to either an annulus or a punctured disk. If $s>-1$, the latter can only occur for $t=\infty$, in which case the corresponding curves accumulate to $(0:1:i:0)\in\mathcal{O}^{(2)}$; if $t<\infty$, it occurs only for the last two curves provided $s=-1$, and in this case they accumulate to $(0,0,-i)\in\mathcal{O}_6$.

It now follows, as before, that either $s=-1$ or $t=\infty$. If $s=-1$ we can extend f to a biholomorphic map between M' and $\hat{\mathfrak{D}}^{(1)}_t$ (see (5.73)) that takes O onto $\mathcal{O}_6$. This is impossible since O is a complex curve in M and $\mathcal{O}_6$ is totally real in $\hat{\mathfrak{D}}^{(1)}_t$. Hence $t=\infty$, and we can extend f to a biholomorphic map between $\hat{M}$ and $\mathfrak{D}^{(1)}_s$ (see (5.73)). It is straightforward to see that O is the only codimension 2 orbit in M, and thus M is holomorphically equivalent to $\mathfrak{D}_s^{(1)}$, which is a manifold listed in (viii) of the theorem.

Next, the case when M' is equivalent to $D_{s,t}^{(2)}$ is treated as the preceding one. Here we parametrize maximal compact subgroups of $\mathcal{R}^{(1)}$ by points in $\mathcal{O}^{(2)}$, and for the point $(0:1:i:0) \in \mathcal{O}^{(2)}$ the corresponding family of complex curves consists of sets constructed as family (5.116), where the curves appearing on the left must be intersected with $D_{s,t}^{(2)}$ rather than $\mathfrak{D}_{s,t}^{(1)}$ (note, however, that the second last intersection is empty). As above, we obtain that $t = \infty$ and that M is holomorphically equivalent to $D_s^{(2)}$ (see (5.68)). We now recall that $D_1^{(2)}$ is equivalent to Δ^2 (see **(11)(c)**), and, excluding the value $s = 1$, obtain (vi) of the theorem.

We now assume that M' is holomorphically equivalent to one of the n-sheeted covers, for $n \geq 2$, of the previously considered possibilities: $\Omega_{s,t}^{(n)}$ (the cover of $\Omega_{s,t}$) with $-1 \leq s < t \leq 1$ – see (5.55); $D_{s,t}^{(n)}$ (the cover of $D_{s,t}$) with $1 \leq s < t \leq \infty$, where n is odd – see **(10)**; $D_{s,t}^{(2n)}$ (the cover of $D_{s,t}^{(2)}$) with $1 \leq s < t \leq \infty$ – see (5.60); $\mathfrak{D}_{s,t}^{(n)}$ (the cover of $\mathfrak{D}_{s,t}^{(1)}$) with $-1 \leq s < 1 < t \leq \infty$, where $s = -1$ and $t = \infty$ do not hold simultaneously – see **(11)(d)**. We will now formulate a number of useful properties that hold for the covers. These properties (that we hereafter refer to as Properties (P)) follow from the explicit construction of the covers in **(10)**, **(11)**.

Let S be one of $\Omega_{s,t}$, $D_{s,t}$, $D_{s,t}^{(2)}$, $\mathfrak{D}_{s,t}^{(1)}$ and let $S^{(n)}$ be the corresponding n-sheeted cover of S (for $S = D_{s,t}$ we assume that n is odd). Let $H := G(S)$ and $H^{(n)} := G\left(S^{(n)}\right)$. Then we have:

(a) the group $H^{(n)}$ consists of all lifts from S to $S^{(n)}$ of all elements of H, and the natural projection $\pi : H^{(n)} \to H$ is a Lie group homomorphism and realizes $H^{(n)}$ as an n-sheeted cover of H;

(b) it follows from (a) that for every maximal compact subgroup $K_0 \subset H$ (all such subgroups are isomorphic to U_1) the subgroup $\pi^{-1}(K_0)$ is maximal compact in $H^{(n)}$, and all maximal compact subgroups of $H^{(n)}$ are obtained in this way;

(c) for every maximal compact subgroup $K \subset H^{(n)}$ the family of all K-invariant complex curves in $S^{(n)}$ consists of the lifts from S to $S^{(n)}$ of all $\pi(K)$-invariant complex curves in S, where every connected $\pi(K)$-invariant curve C is lifted to a unique connected K-invariant curve $C^{(n)}$ (in particular, $C^{(n)}$ covers C in an n-to-1 fashion);

(d) if S is one of $D_{s,t}$, $D_{s,t}^{(2)}$, $\mathfrak{D}_{s,t}^{(1)}$, then every maximal compact subgroup $K \subset H^{(n)}$ is the isotropy subgroup – with respect to the $H^{(n)}$-action – of a unique point in $\mathcal{O}^{(n)}$ (see **(11)(b)**) in the first case and a unique point in $\mathcal{O}^{(2n)}$ (see (5.66)) in each of the other two cases; every K-invariant closed complex curve in $S^{(n)}$ equivalent to a punctured disk accumulates to this point (provided, for $S = \mathfrak{D}_{s,t}^{(1)}$, we assume that $s > -1$).

Properties (P) yield that if M' is equivalent to either $D^{(n)}_{s,t}$ for odd n or $D^{(2n)}_{s,t}$ for $n \geq 2$, then $t = \infty$ and M is holomorphically equivalent to either $D^{(n)}_{s}$ (see (5.69)) or $D^{(2n)}_{s}$ (see (5.68)), respectively; this gives (vii) of the theorem.

Suppose now that M' is equivalent to $\Omega^{(n)}_{s,t}$ by means of a map f. Then Properties (P) imply that $s = -1$. Recall that $\Psi_\nu \circ \Phi^{(n)}_\nu : \Omega^{(n)}_{-1,t} \to \Omega_{-1,t}$ is an n-to-1 covering map (see **(10)**). Consider the composition $\tilde{f} := \Psi_\nu \circ \Phi^{(n)}_\nu \circ f$. This is an n-to-1 covering map from M' onto $\Omega_{-1,t}$ satisfying (3.16) for all $g \in G(M)$, $q \in M'$, where $\varphi : G(M) \to \mathcal{R}_\nu$ is an n-to-1 covering homomorphism. Fix $p_0 \in O$. Since $K_0 := \varphi(I_{p_0})$ is a maximal compact subgroup of $\mathcal{R}_\nu$, there is a unique $q_0 \in \mathcal{O}_5$ such that K_0 is the isotropy subgroup of q_0 under the $\mathcal{R}_\nu$-action on Ω_1. We now define $\tilde{f}(p_0) := q_0$. Thus, we have extended $\tilde{f}$ to an equivariant map from $\hat{M}$ onto Ω_t that takes O onto $\mathcal{O}_5$. As before, it can be shown that $\tilde{f}$ is holomorphic on $\hat{M}$. However, $\mathcal{O}_5$ is totally real in $\Omega_{-1,t}$ and therefore M' cannot in fact be equivalent to $\Omega^{(n)}_{s,t}$.

Let M' be equivalent to $\mathfrak{D}^{(n)}_{s,t}$, and let f be an equivalence map. In this case Properties (P) imply that either $s = -1$ or $t = \infty$. If $s = -1$, arguing as in the preceding paragraph, we extend the map $\tilde{f} := \Phi^{(n)} \circ f$ to a holomorphic map from $\hat{M}$ onto $\hat{\mathfrak{D}}^{(1)}_t$ that takes O onto $\mathcal{O}_6$. As before, this is impossible since $\mathcal{O}_6$ is totally real in $\hat{\mathfrak{D}}^{(1)}_t$, and therefore we in fact have $t = \infty$. In this case Properties (P) yield that M is holomorphically equivalent to $\mathfrak{D}^{(n)}_s$, and we have obtained (viii) of the theorem.

We now assume that every codimension 2 orbit in M is totally real. We will go again through all the possibilities for the group $G(M)$ listed in Proposition 5.3, paying attention to constraints imposed on $G(M)$ by this condition. In what follows O denotes a totally real orbit in M. In case (E) with $G(M)$ isomorphic to $G_{\varepsilon'_\alpha}$ (see (5.112)) we obtain, as before, that $I_p = I := \varphi^{-1}(J^{\varepsilon'_\alpha})$ for every $p \in O$, and thus I_p acts trivially on $O(p)$ for every $p \in O$ which contradicts (iv) of Proposition 4.2. A similar argument gives a contradiction in case (G). In case (E) with $G(M)$ isomorphic to $\mathfrak{V}_\alpha$ the argument given above for the case of complex curve orbits shows that f extends to a biholomorphic map between $\hat{M}$ and either $\mathfrak{E}'_{\alpha,s}$ (see (5.113)) or domain (5.114), with either $f(O) = \mathcal{O}_7$ or $f(O) = \mathcal{O}_8$, respectively, which is impossible, since O is totally real, whereas $\mathcal{O}_6$, $\mathcal{O}_7$ are complex curves. Next, in subcase (l') of case (A) the group $G(M)$ is isomorphic to SU_2, which implies that M is holomorphically equivalent to one of the manifolds listed in [IKru2] (see Sect. 6.3). However, none of the manifolds on the list has a totally real orbit. Therefore, it remains to consider subcases (j), (j"), (l), (l"), (m), (m"), (n), (n") of case (A), and case (C).

We start with case (C). In this situation $G(M)$ is isomorphic to SU_2, if m is odd and to $SU_2/\{\pm\mathrm{id}\}$, if m is even. To rule out the case of odd m we again use the result of [IKru2] (see Sect. 6.3). We now assume that m is even. In this case M' is holomorphically equivalent to $S_{m,s,t}$ (see (5.85)), with

$0 \le s < t < \infty$, by means of a map f that satisfies (3.16) for all $g \in G(M)$, $q \in M'$ and some isomorphism $\varphi : G(M) \to SU_2/\{\pm \mathrm{id}\}$.

Fix p_0 in O and consider the connected compact 1-dimensional subgroup $\varphi(I^0_{p_0}) \subset SU_2/\{\pm \mathrm{id}\}$. It then follows that $\varphi(I^0_{p_0})$ is conjugate in $SU_2/\{\pm \mathrm{id}\}$ to the subgroup $J^{\mathcal{L}}$ that consists of all elements of the form

$$\begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix} \{\pm \mathrm{id}\},$$

where $\psi \in \mathbb{R}$ (see e.g. Lemma 2.1 of [IKru1]). Suppose that p_0 is chosen so that $\varphi(I^0_{p_0}) = J^{\mathcal{L}}$. Let C_{p_0} be a connected $I^0_{p_0}$-invariant complex curve in M that intersects O transversally at p_0. Then $f(C_{p_0} \setminus \{p_0\})$ is a connected $J^{\mathcal{L}}$-invariant complex curve in $S_{m,s,t}$. It is straightforward to see that every connected $J^{\mathcal{L}}$-invariant complex curve in $S_{m,s,t}$ extends to a closed curve equivalent to either an annulus or a punctured disk. The only closed connected $J^{\mathcal{L}}$-invariant curves in $S_{m,s,t}$ that can be equivalent to a punctured disk (which only occurs for $s = 0$) are

$$\Big(\{z = 0\}/\mathbb{Z}_m\Big) \cap S_{m,s,t}$$

and

$$\Big(\{w = 0\}/\mathbb{Z}_m\Big) \cap S_{m,s,t}.$$

Therefore, the curve $f(C_{p_0} \setminus \{p_0\})$ extends to one of these curves, and we have $s = 0$.

Let B_t be the ball of radius t in $\mathbb{C}^2$ and $\widehat{B_t}$ its *blow-up* at the origin, i.e.

$$\widehat{B_t} := \Big\{ \Big[(z,w),(\xi:\zeta)\Big] \in B_t \times \mathbb{CP}^1 : z\zeta = w\xi \Big\}, \tag{5.117}$$

where $(\xi : \zeta)$ are the homogeneous coordinates in $\mathbb{CP}^1$. We define an action of U_2 on $\widehat{B_t}$ as follows: for $g \in U_2$ and $\Big[(z,w),(\xi:\zeta)\Big] \in \widehat{B_t}$ set

$$g\Big[(z,w),(\xi:\zeta)\Big] := \Big[g(z,w), g(\xi:\zeta)\Big],$$

where in the right-hand side we use the standard actions of U_2 on $\mathbb{C}^2$ and $\mathbb{CP}^1$. Next, we denote by $\widehat{B_t}/\mathbb{Z}_m$ the quotient of $\widehat{B_t}$ by the equivalence relation $\Big[(z,w),(\xi:\zeta)\Big] \sim e^{\frac{2\pi i}{m}}\Big[(z,w),(\xi:\zeta)\Big]$. Let $\Big\{\Big[(z,w),(\xi:\zeta)\Big]\Big\} \in \widehat{B_t}/\mathbb{Z}_m$ be the equivalence class of $\Big[(z,w),(\xi:\zeta)\Big] \in \widehat{B_t}$. We now define in a natural way an action of $SU_2/\{\pm \mathrm{id}\}$ on $\widehat{B_t}/\mathbb{Z}_m$: for $\Big\{\Big[(z,w),(\xi:\zeta)\Big]\Big\} \in \widehat{B_t}/\mathbb{Z}_m$ and $g\{\pm \mathrm{id}\} \in SU_2/\{\pm \mathrm{id}\}$ we set

$$g\{\pm \mathrm{id}\}\Big\{\Big[(z,w),(\xi:\zeta)\Big]\Big\} := \Big\{g\Big[(z,w),(\xi:\zeta)\Big]\Big\}.$$

The points $\Big\{\Big[(0,0),(\xi:\zeta)\Big]\Big\}$ form an $SU_2/\{\pm \mathrm{id}\}$-orbit that we denote by $\mathcal{O}_9$; this is a complex curve equivalent to $\mathbb{CP}^1$. Everywhere below we identify $S_{m,0,t}$ with $\widehat{B_t}/\mathbb{Z}_m \setminus \mathcal{O}_9$.

For $q_0 \in \mathcal{O}_9$ let $J^{\mathcal{L}}_{q_0}$ be the isotropy subgroup under the action of $SU_2/\{\pm\mathrm{id}\}$. It is straightforward to see that every subgroup $J^{\mathcal{L}}_{q_0}$ is conjugate to $J^{\mathcal{L}}$ in $SU_2/\{\pm\mathrm{id}\}$ and that for every q_0 there is exactly one $q_0' \in \mathcal{O}_9$, $q_0' \neq q_0$, such that $J^{\mathcal{L}}_{q_0} = J^{\mathcal{L}}_{q_0'}$ (for example, $J^{\mathcal{L}}$ is the isotropy subgroup of each of $\Big\{\Big[(0,0),(1:0)\Big]\Big\}$ and $\Big\{\Big[(0,0),(0:1)\Big]\Big\}$). Fix $q_0 \in \mathcal{O}_9$ and let $p_0 \in O$ be such that $\varphi(I^0_{p_0}) = J^{\mathcal{L}}_{q_0}$. As we noted at the beginning of the proof of the theorem, there are exactly two connected $I^0_{p_0}$-invariant complex curves C_{p_0} and $\tilde{C}_{p_0}$ near p_0 that intersect O at p_0 transversally. The curves $f(C_{p_0} \setminus \{p_0\})$ and $f(\tilde{C}_{p_0} \setminus \{p_0\})$ extend to the two distinct closed $J^{\mathcal{L}}_{q_0}$-invariant complex curves in $\widehat{B_t}/\mathbb{Z}_m \setminus \mathcal{O}_9$ that are equivalent to a punctured disk. Since there are no other closed $J^{\mathcal{L}}_{q_0}$-invariant complex curves in $\widehat{B_t}/\mathbb{Z}_m \setminus \mathcal{O}_9$ equivalent to a punctured disk, it follows that $I^0_{p_0'} \neq I^0_{p_0}$ for every $p_0' \in O$, $p_0' \neq p_0$.

Observe that if $q, q' \in \mathcal{O}_9$, $q \neq q'$, are such that $J^{\mathcal{L}}_q = J^{\mathcal{L}}_{q'} =: J$, then one of the J-invariant complex curves equivalent to a punctured disk accumulates to q and the other to q'. Therefore, we can extend $\mathfrak{F} := f^{-1}$ to a map from $\widehat{B_t}/\mathbb{Z}_m$ onto $\hat{M}$ by setting $\mathfrak{F}(q_0) := p_0$, where $q_0 \in \mathcal{O}_9$ and $p_0 \in O$ are related as indicated above (hence $\mathfrak{F}$ is 2-to-1 on $\mathcal{O}_9$). As before, it can be shown that $\mathfrak{F}$ is continuous on $\widehat{B_t}/\mathbb{Z}_m$ and thus is holomorphic there. However, $\mathfrak{F}$ maps the complex curve $\mathcal{O}_9 \subset \widehat{B_t}/\mathbb{Z}_m$ onto the totally real submanifold $O \subset \hat{M}$, which is impossible. Hence, M' cannot be equivalent to $S_{m,s,t}$.

We now consider the remaining subcases of case (A). In subcase (j) the manifold M' is holomorphically equivalent to $\mathfrak{S}_{s,t}$ for $0 \leq s < t < \infty$ (see (5.9)). The group $\mathcal{R}_\chi$ (see (5.10)) acts on $\mathbb{C}^2$ with the only codimension 2 orbit $\mathcal{O}_2$ (see (5.13)). The isotropy subgroup of a point $(iy_0, iv_0) \in \mathcal{O}_2$ is the group $J^\chi_{(y_0,v_0)}$ of all transformations of the form (5.10) with $\beta = y_0 - \cos\psi \cdot y_0 - \sin\psi \cdot v_0$, $\gamma = v_0 + \sin\psi \cdot y_0 - \cos\psi \cdot v_0$, where

$$A = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix},$$

with $\psi \in \mathbb{R}$. Note that these subgroups are maximal compact in $\mathcal{R}_\chi$ (which implies that I_p is connected for every $p \in O$), and the isotropy subgroups of distinct points in $\mathcal{O}_2$ do not coincide.

We now argue as in the second part of case (E) for complex curve orbits. There is a family $\mathcal{F}^{\mathfrak{S}}_{(y_0,v_0)}$ of connected closed complex curves in $\mathfrak{S}_{s,t}$ invariant under the $J^\chi_{(y_0,v_0)}$-action, such that every connected $J^\chi_{(y_0,v_0)}$-invariant complex curve in $\mathfrak{S}_{s,t}$ extends to a curve from $\mathcal{F}^{\mathfrak{S}}_{(y_0,v_0)}$. The family $\mathcal{F}^{\mathfrak{S}}_{(0,0)}$ consists of the connected components of non-empty sets analogous to (5.115), where the sets on the left must be intersected with $\mathfrak{S}_{s,t}$ rather than $\Omega_{s,t}$. Among the curves from $\mathcal{F}^{\mathfrak{S}}_{(0,0)}$, only the last two can be equivalent to a punctured disk. This occurs only for $s = 0$, in which case the curves accumulate to $(0,0) \in \mathcal{O}_2$. Arguing as before, we can now construct a biholomorphic map between M

and $\mathfrak{S}_t$ (see (5.12)). Clearly, $\mathfrak{S}_t$ is equivalent to $\mathfrak{S}_1$, and we have obtained (i) of the theorem.

Consider subcase (j"). In this situation M' is holomorphically equivalent to the n-sheeted cover $\mathfrak{S}_{s,t}^{(n)}$ of $\mathfrak{S}_{s,t}$ for $0 \le s < t < \infty$, $n \ge 2$ (see (5.15)), by means of a map f. From the explicit construction of the covers in **(4)** it follows that Properties (P) hold for $S = \mathfrak{S}_{s,t}$. Let $\tilde{f} := \Phi_\chi^{(n)} \circ f$, where $\Phi_\chi^{(n)} : \mathfrak{S}_{s,t}^{(n)} \to \mathfrak{S}_{s,t}$ is the n-to-1 covering map defined in (5.14). Arguing as in the case of complex curve orbits when M' was assumed to be equivalent to $\Omega_{s,t}^{(n)}$, we extend $\tilde{f}$ to a holomorphic map from $\hat{M} = M' \cup O$ onto $\mathfrak{S}_t$, that takes O onto $\mathcal{O}_2$.

Suppose that the differential of $\tilde{f}$ is degenerate at a point in O. Then, since $\tilde{f}$ satisfies (3.16), its differential degenerates everywhere on O. Since O is totally real, it follows that the differential of $\tilde{f}$ is degenerate everywhere in $\hat{M}$. This is impossible since $\tilde{f}$ is a covering map on M', and thus $\tilde{f}$ is in fact non-degenerate at every point of O. Hence, for every $p \in O$ there exists a neighborhood of p in which $\tilde{f}$ is biholomorphic. Fix $p_0 \in O$ and let C_{p_0} be a connected I_{p_0}-invariant complex curve intersecting O at p_0 transversally (observe that I_{p_0} is connected). Then it follows from (c) of Properties (P) that $f(C_{p_0} \setminus \{p_0\})$ covers $\tilde{f}(C_{p_0} \setminus \{p_0\})$ in an n-to-1 fashion, and hence $\tilde{f}$ cannot be biholomorphic in any neighborhood of p_0. This contradiction yields that M' cannot be equivalent to $\mathfrak{S}_{s,t}^{(n)}$.

Consider subcase (l). In this situation M' is holomorphically equivalent to $E_{s,t}$ for $1 \le s < t < \infty$ (see (5.23)). The group R_μ (see (5.24)) acts on $\mathbb{CP}^2$ with the totally real orbit $\mathcal{O}_3$ (see (5.27)). Every connected 1-dimensional compact subgroup of $\mathcal{R}_\mu$ is conjugate in $\mathcal{R}_\mu$ to the subgroup J^μ that consists of all matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{pmatrix}, \tag{5.118}$$

where $\psi \in \mathbb{R}$ (this follows, for instance, from Lemma 2.1 of [IKru1]). It is straightforward to see that the isotropy subgroup J_q^μ of a point $q \in \mathcal{O}_3$ under the $\mathcal{R}_\mu$-action is conjugate to J^μ (note that $J^\mu = J_{(1:0:0)}^\mu$) and that the isotropy subgroups of distinct points do not coincide.

There is a family $\mathcal{F}_q^E$ of connected closed complex curves in $E_{s,t}$ invariant under the J_q^μ-action, such that every connected J_q^μ-invariant complex curve in $E_{s,t}$ extends to a curve from $\mathcal{F}_q^E$. The family $\mathcal{F}_{(1:0:0)}^E$ consists of the connected components of non-empty sets of the form

$$\begin{aligned} &\big\{(\zeta : z : w) \in \mathbb{CP}^2 : z^2 + w^2 = \rho\zeta^2\big\} \cap E_{s,t}, \\ &\big\{(\zeta : z : w) \in \mathbb{CP}^2 : z = iw\big\} \cap E_{s,t}, \\ &\big\{(\zeta : z : w) \in \mathbb{CP}^2 : z = -iw\big\} \cap E_{s,t}, \end{aligned}$$

where $\rho \in \mathbb{C}^*$. Among the curves from $\mathcal{F}_{(1:0:0)}^E$, only the last two can be equivalent to a punctured disk. This occurs only for $s = 1$, in which case the curves accumulate to $(1:0:0) \in \mathcal{O}_3$. Arguing as before, we can now construct a biholomorphic map between M and E_t (see (5.26)), which gives (ii) of the theorem.

Further, in subcase (l") M' is holomorphically equivalent to $E^{(2)}_{s,t}$ for some $1 \le s < t < \infty$ (see (5.30)). Let f be an equivalence map that satisfies (3.16) for all $g \in G(M)$, $q \in M'$ and some isomorphism $\varphi : G(M) \to \mathcal{R}^{(2)}_\mu$ (see (5.31)). The group $\mathcal{R}^{(2)}_\mu$ acts on $\mathcal{Q}_+$ (see (5.28)) with the totally real orbit $\mathcal{O}_4$ (see (5.36)). All 1-dimensional compact subgroups are described as in subcase (l) – see (5.118). The isotropy subgroup $J^{\mu^{(2)}}_q$ of a point $q \in \mathcal{O}_4$ under the $\mathcal{R}^{(2)}_\mu$-action is conjugate to J^μ, and for every $q \in \mathcal{O}_4$ there exists exactly one $q' \in \mathcal{O}_4$, $q' \ne q$, for which $J^{\mu^{(2)}}_q = J^{\mu^{(2)}}_{q'}$ (note that $q' = -q$ and $J^\mu = J^{\mu^{(2)}}_{(\pm 1,0,0)}$).

Again, there is a family $\mathcal{F}^{E^{(2)}}_q$ of connected closed complex curves in $E^{(2)}_{s,t}$ invariant under the $J^{\mu^{(2)}}_q$-action, such that every connected $J^{\mu^{(2)}}_q$-invariant complex curve in $E^{(2)}_{s,t}$ extends to a curve from $\mathcal{F}^{E^{(2)}}_q$. The family $\mathcal{F}^{E^{(2)}}_{(\pm 1,0,0)}$ consists of the connected components of non-empty sets of the form

$$\begin{aligned}
&\left\{(z_1,z_2,z_3) \in \mathbb{C}^3 : z_2^2 + z_3^2 = \rho z_1^2\right\} \cap E^{(2)}_{s,t},\\
\mathfrak{C}_1 &:= \left\{(z_1,z_2,z_3) \in \mathbb{C}^3 : z_1 = 1,\ z_2 = iz_3\right\} \cap E^{(2)}_{s,t},\\
\mathfrak{C}_2 &:= \left\{(z_1,z_2,z_3) \in \mathbb{C}^3 : z_1 = 1,\ z_2 = -iz_3\right\} \cap E^{(2)}_{s,t},\\
\mathfrak{C}_3 &:= \left\{(z_1,z_2,z_3) \in \mathbb{C}^3 : z_1 = -1,\ z_2 = iz_3\right\} \cap E^{(2)}_{s,t},\\
\mathfrak{C}_4 &:= \left\{(z_1,z_2,z_3) \in \mathbb{C}^3 : z_1 = -1,\ z_2 = -iz_3\right\} \cap E^{(2)}_{s,t},
\end{aligned}$$

where $\rho \in \mathbb{C}^*$. Among the curves from $\mathcal{F}^{E^{(2)}}_{(\pm 1,0,0)}$, only $\mathfrak{C}_j$ can be equivalent to a punctured disk, which occurs only for $s = 1$. It then follows that $s = 1$, and in this case $\mathfrak{C}_1$, $\mathfrak{C}_2$ accumulate to $(1,0,0) \in \mathcal{O}_4$, while $\mathfrak{C}_3$, $\mathfrak{C}_4$ accumulate to $(-1,0,0) \in \mathcal{O}_4$.

Fix $p_0 \in O$ and let $q_0 \in \mathcal{O}_4$ be such that $\varphi(I^0_{p_0}) = J^{\mu^{(2)}}_{q_0}$ and such that, for a $I^0_{p_0}$-invariant complex curve C_{p_0} intersecting O at p_0 transversally, the curve $f\left(C_{p_0} \setminus \{p_0\}\right)$ extends to a complex curve from $\mathcal{F}^{E^{(2)}}_{q_0}$ that accumulates to q_0. We extend $\mathfrak{F} := f^{-1}$ to a map from $E^{(2)}_t$ (see (5.35)) onto $\hat{M} = M' \cup O$ that takes $\mathcal{O}_4$ onto O. Define $\mathfrak{F}(q_0) := p_0$ and for any $h \in \mathcal{R}^{(2)}_\mu$ set $\mathfrak{F}(hq_0) := \varphi^{-1}(h)p_0$. Since $\varphi^{-1}\left(J^{\mu^{(2)}}_{q_0}\right) \subset I_{p_0}$, this map is well-defined. Furthermore, the extended map satisfies (3.17) for all $h \in \mathcal{R}^{(2)}_\mu$, $q \in E^{(2)}_t$, and for every $q \in \mathcal{O}_4$ there exists a $J^{\mu^{(2)}}_q$-invariant complex curve $\mathfrak{C}$ in $E^{(2)}_t$ that intersects $\mathcal{O}_4$ at q transversally and such that $\mathfrak{F}\left(\mathfrak{C} \setminus \{q\}\right)$ is an $I^0_{\mathfrak{F}(q)}$-invariant complex curve that accumulates to $\mathfrak{F}(q)$. Arguing as in the second part of case (E) for complex curve orbits, we now obtain that $\mathfrak{F}$ is holomorphic on $E^{(2)}_t$. Further, as in subcase (j") above, we see that $\mathfrak{F}$ is locally biholomorphic in a neighborhood of every point in $\mathcal{O}_4$.

We will now show that $\mathfrak{F}$ is 1-to-1 on $\mathcal{O}_4$. Suppose that for some $q, q' \in \mathcal{O}_4$, $q \ne q'$, we have $\mathfrak{F}(q) = \mathfrak{F}(q') = p$ for some $p \in O$. Since $\mathfrak{F}$ satisfies (3.17), we have $J_q^{\mu^{(2)}} = J_{q'}^{\mu^{(2)}} = \varphi(I_p^0)$, and therefore $q' = -q$. Consider the four $J_q^{\mu^{(2)}}$-invariant connected complex curves in $E_{1,t}^{(2)}$ equivalent to a punctured disk; a pair of these curves accumulates to q, while the other pair accumulates to $-q$. The curves are mapped by $\mathfrak{F}$ into four distinct I_p^0-invariant complex curves in M' whose extensions in $\hat{M}$ intersect O transversally at p. However, as we noted at the beginning of the proof of the theorem, there are exactly two I_p^0-invariant complex curves near p that intersect O transversally at p. This contradiction yields that $\mathfrak{F}$ is a biholomorphic map from $E_t^{(2)}$ onto $\hat{M}$. It can be now shown, as before, that O is the only codimension 2 orbit in M, which gives that M is holomorphically equivalent to $E_t^{(2)}$, and we have obtained (iii) of the theorem.

It now remains to consider subcases (m), (m"), (n), (n"). We will proceed as in the situation when a complex curve orbit was assumed to be present in M. If M' is equivalent to one of $D_{s,t}$, $D_{s,t}^{(n)}$ for $n \geq 2$, $\mathfrak{D}_{s,t}^{(n)}$ for $n \geq 1$ (where in the last case we assume that $s > -1$), we obtain a contradiction since O is totally real in M whereas $\mathcal{O}$ and $\mathcal{O}^{(n)}$ for $n \geq 2$ are complex curves in the corresponding manifolds. Further, if M' is equivalent to $\Omega_{s,t}$, we obtain that $s = -1$ and M is holomorphically equivalent to Ω_t. Recalling that Ω_1 is equivalent to Δ^2 (see **(11)(c)**) and excluding the value $t = 1$, we obtain (iv) of the theorem. Next, If M' is equivalent to $\mathfrak{D}_{-1,t}^{(1)}$, then M is equivalent to $\hat{\mathfrak{D}}_t^{(1)}$, which are the manifolds in (ix) of the theorem.

Suppose now that M' is equivalent to $\Omega_{s,t}^{(n)}$ for some $-1 \leq s < t \leq \infty$. In this case we obtain a holomorphic map $\tilde{f}$ from $\hat{M}$ onto Ω_t that takes O onto $\mathcal{O}_5$ and such that $\tilde{f}|_{M'}$ is an n-to-1 covering map from M' onto $\Omega_{-1,t}$. Now, arguing as in subcase (j"), we obtain that the differential of $\tilde{f}$ is non-degenerate at every point of O which leads to a contradiction. Finally, a similar argument leads to a contradiction if M' is equivalent to $\mathfrak{D}_{-1,t}^{(n)}$ for $n \geq 2$.

The proof of Theorem 5.7 is complete. ∎

6

Proper Actions

As we have seen, the results presented in Chaps. 1–5 rely to a great extent on the properness of the $\text{Aut}(M)$-action on a hyperbolic manifold M. Accordingly, the majority of the results admit (or should admit) generalizations to the case of proper effective actions by holomorphic transformations on not necessarily hyperbolic manifolds. In this concluding part of the book we briefly discuss ways in which generalizations of this kind can be obtained. Apart from a few special cases, we do not give complete classifications and detailed proofs here, they will appear in our future work (see e.g. [I6]).

6.1 General Remarks

If G is a topological group acting properly and effectively by diffeomorphisms on a smooth manifold N (recall that the action is given by a continuous injective homomorphism $\Phi : G \to \text{Diff}(N)$), then G is locally compact. The results of [BM1], [BM2], [MZ] therefore imply that G is a Lie transformation group, and we denote by d_G its dimension. Furthermore, the topology of G coincides with the pull-back of the compact-open topology by means of Φ (see e.g. [Bi]), that is, G is isomorphic to $\Phi(G)$ as a topological group.

Suppose now that G acts on a connected complex manifold M by holomorphic transformations (in this case Φ maps G into $\text{Aut}(M)$). Without loss of generality we assume that G is realized as a closed subgroup of $\text{Aut}(M)$ (that is, we identify G with $\Phi(G)$). For $p \in M$ let $O_G(p) := \{gp : g \in G\}$ be the G-orbit of p, let $I_{G,p} := \{g \in G : gp = p\}$ be the isotropy subgroup in G, let $L_{G,p} := \{dg_p : g \in I_{G,p}\}$ be the corresponding linear isotropy subgroup in $GL(\mathbb{C}, T_p(M))$, and let $\alpha_{G,p} : g \mapsto dg_p$ be the isotropy representation of $I_{G,p}$. Also, let n, as before, denote the complex dimension of M. Then, arguing as in Sect. 1.2 (see (1.3)), we obtain

$$d_G \leq n^2 + 2n.$$

It is natural to formulate the classification problem for complex manifolds equipped with group actions as the problem of describing pairs (M, G) as specified above, rather than just manifolds themselves. That is to say, if we have shown that M is holomorphically equivalent to a model manifold M_0, it is also desirable to find a biholomorphism between M and M_0 that transforms G into a model group $G_0 \subset \mathrm{Aut}(M_0)$ whose action on M_0 is explicitly given. With this in mind, we will now discuss ways in which classification results parallel to those in Chaps. 1–5 can be obtained, assuming that $d_G \geq n^2 - 1$.

First of all, Theorem 1.1 can be generalized by showing that, apart from the unit ball $\mathbb{B}^n$, there are exactly two – up to holomorphic equivalence – complex manifolds that admit a proper action of a group G with $d_G = n^2 + 2n$. Note that if $d_G = n^2 + 2n$, then G acts on M transitively (in which case we say that M is *G-homogeneous* or, when doing so does not lead to confusion, simply homogeneous).

We denote by $G(\mathbb{C}^n)$ the group of all holomorphic automorphisms of $\mathbb{C}^n$ of the form

$$z \mapsto Uz + a, \tag{6.1}$$

where $U \in U_n$, $a \in \mathbb{C}^n$. Clearly, $G(\mathbb{C}^n)$ is isomorphic to $U_n \ltimes \mathbb{C}^n$, has dimension $n^2 + 2n$ and acts properly on $\mathbb{C}^n$. Further, we denote by $G(\mathbb{CP}^n)$ the group of all holomorphic automorphisms of $\mathbb{CP}^n$ of the form

$$\zeta \mapsto U\zeta,$$

where ζ is a point in $\mathbb{CP}^n$ given in homogeneous coordinates, and $U \in SU_{n+1}$. The group $G(\mathbb{CP}^n)$ is isomorphic to $PSU_{n+1} := SU_{n+1}/Z$, where Z is the center of SU_{n+1}, has dimension $n^2 + 2n$ and acts properly on $\mathbb{CP}^n$. Observe also that $G(\mathbb{CP}^n)$ is a maximal compact subgroup of the complex Lie group $\mathrm{Aut}(\mathbb{CP}^n) \simeq PSL_{n+1}(\mathbb{C}) := SL_{n+1}(\mathbb{C})/Z$.

We are now ready to formulate a generalization of Theorem 1.1.

Theorem 6.1. ([Ka]) *Let M be a connected complex manifold of dimension n. Suppose that $G \subset Aut(M)$ is a subgroup whose action on M is proper, and $d_G = n^2 + 2n$. Then M is holomorphically equivalent to one of $\mathbb{B}^n$, $\mathbb{C}^n$, $\mathbb{CP}^n$, and an equivalence map can be chosen so that it transforms G into one of the groups $Aut(\mathbb{B}^n)$, $G(\mathbb{C}^n)$, $G(\mathbb{CP}^n)$, respectively.*

Proof: Fix $p \in M$. As in the proof of Theorem 1.1, we use the fact that $L_{G,p} = U_n$ acts transitively on real directions in $T_p(M)$. By Theorems 1.7 and 1.11, this implies that M is holomorphically equivalent to one of $\mathbb{B}^n$, $\mathbb{C}^n$, $\mathbb{CP}^n$. Let f denote an equivalence map. If M is equivalent to $\mathbb{B}^n$, then f clearly transforms G into $\mathrm{Aut}(\mathbb{B}^n)$. If M is equivalent to $\mathbb{CP}^n$, then M is compact, and, since $I_{G,p}$ is compact, it follows that G is compact. Therefore, f transforms G into a maximal compact subgroup of $\mathrm{Aut}(\mathbb{CP}^n)$. This subgroup is conjugate to $G(\mathbb{CP}^n)$, and it follows that there exists a biholomorphic map between M and $\mathbb{CP}^n$ that transforms G into $G(\mathbb{CP}^n)$. Finally, if M is equivalent to $\mathbb{C}^n$, the proof of Theorem 1.7 shows that f can be chosen so that it transforms

$I_{G,p}$ into U_n (see Sect. 1.4). Then Satz 4.9 of [Ka] implies that f transforms G into $G(\mathbb{C}^n)$.

The proof is complete. ■

We note that, instead of using Theorems 1.7 and 1.11, in the proof of Theorem 6.1 we could refer to the classification of isotropic Riemannian manifolds as in [Ak] (cf. Remark 1.2).

Next, the proof of Proposition 2.1 yields the following proposition.

Proposition 6.2. ([Ka])*Let M be a connected complex manifold of dimension n. Suppose that $G \subset Aut(M)$ is a subgroup whose action on M is proper, and $d_G > n^2$. Then the action of G on M is transitive.*

Our study of homogeneous hyperbolic manifolds in Chap. 2 relied on the fact that every such manifold is known to be holomorphically equivalent to a Siegel domain of the second kind (see [N], [P-S]), and on the description of the Lie algebra of the automorphism group of a Siegel domain (see [KMO], [Sa]). Unfortunately, there exists no analogous set of models for all connected complex manifolds homogeneous under proper actions. While one may be able to find some models in special cases (if we think of Siegel domains as a set of models associated with $\mathbb{B}^n$, there may be sets of models associated with $\mathbb{C}^n$ and $\mathbb{CP}^n$ as well), it is hardly possible to have ones in full generality. Indeed, there are far too many homogeneous manifolds: for example, every complex Lie group acts properly and effectively on itself by left multiplication. Since no uniform approach to G-homogeneous manifolds seems to exist, one has to deal with them on a case-by-case basis. We will now briefly explain how it can be done for high-dimensional groups. From now on we assume that $n \geq 2$.

Suppose that $n^2 + 3 \leq d_G < n^2 + 2n$. Fix $p \in M$. Then, arguing as in the proof of Theorem 1.3, we see that either $L_{G,p} = U_n$ or $L^0_{G,p} = SU_n$ or, for $n = 4$, $L^0_{G,p}$ is conjugate in U_4 to $e^{i\mathbb{R}}Sp_2$. In each of the cases, $L_{G,p}$ acts transitively on real directions in $T_p(M)$, and therefore Theorems 1.7 and 1.11 imply that M is holomorphically equivalent to one of $\mathbb{B}^n$, $\mathbb{C}^n$, $\mathbb{CP}^n$ (see [IKra2]).

If $L_{G,p} = U_n$, then $d_G = n^2 + 2n$, which is impossible. Suppose that $L^0_{G,p} = SU_n$. Then $d_G = n^2 + 2n - 1$. This is, however, impossible either if M is equivalent to $\mathbb{B}^n$ since the group $\mathrm{Aut}(\mathbb{B}^n)$ does not have codimension 1 subgroups for $n \geq 2$ (see [EaI]). If M is equivalent to $\mathbb{CP}^n$ by means of a map f, then G is compact, and therefore f transforms G into a codimension 1 subgroup of a maximal compact subgroup of $\mathrm{Aut}(\mathbb{CP}^n)$. Hence there exists a biholomorphic map between M and $\mathbb{CP}^n$ that transforms G into a codimension 1 subgroup of $G(\mathbb{CP}^n)$. However, by Lemma 1.4 the group SU_{n+1}/Z does not have closed codimension 1 subgroups.[1] Further, observe that $\mathrm{Aut}(\mathbb{C}^n)$ *does* contain an $(n^2 + 2n - 1)$-dimensional subgroup whose action on $\mathbb{C}^n$ is proper,

[1] As Dmitri Akhiezer pointed out to us, this fact also follows from a more general classical result of S. Lie that can be found in [Lie].

namely, the subgroup $G_1(\mathbb{C}^n) \subset G(\mathbb{C}^n)$ that consists of all maps of the form (6.1) with $U \in SU_n$. In addition, there are disconnected subgroups of $\mathrm{Aut}(\mathbb{C}^n)$ acting properly on $\mathbb{C}^n$ whose connected identity component coincides with $G_1(\mathbb{C}^n)$, and one can show that a biholomorphic equivalence f between M and $\mathbb{C}^n$ can be chosen so that it transforms G^0 into $G_1(\mathbb{C}^n)$ (note that it follows from the construction given in the proof of Theorem 1.7 that f can be chosen so that it transforms $I^0_{G,p}$ into SU_n – see Sect. 1.4).

Similarly, if $n = 4$ and $L^0_{G,p}$ is conjugate to $e^{i\mathbb{R}}Sp_2$ in U_4, one needs to investigate for each of $\mathbb{B}^4$, $\mathbb{C}^4$, $\mathbb{CP}^4$ the question of the existence of a corresponding 19-dimensional subgroup of $\mathrm{Aut}(\mathbb{B}^4)$, $\mathrm{Aut}(\mathbb{C}^4)$, $\mathrm{Aut}(\mathbb{CP}^4)$ acting properly on the respective manifold. Regarding this, we note here that if M is equivalent to $\mathbb{CP}^4$, then any equivalence map transforms G into a 19-dimensional subgroup of a maximal compact subgroup of $\mathrm{Aut}(\mathbb{CP}^4)$. Hence there exists a biholomorphic map between M and $\mathbb{CP}^4$ that transforms G into a 19-dimensional subgroup of $G(\mathbb{CP}^4)$. However, by Lemma 1.4 the group SU_5/Z does not have closed 19-dimensional subgroups. Observe also that $\mathrm{Aut}(\mathbb{C}^4)$ has a 19-dimensional subgroup whose action on $\mathbb{C}^4$ is proper, namely, the subgroup $G_2(\mathbb{C}^4)$ that consists of all maps of the form (6.1) for $n = 4$ with $U \in e^{i\mathbb{R}}Sp_2$. It is possible to show that M is indeed equivalent to $\mathbb{C}^4$ and that an equivalence map can be chosen so that it transforms G^0 into $G_2(\mathbb{C}^4)$.

Next, let $d_G = n^2 + 2$ and fix $p \in M$. In this case $L_{G,p}$ has dimension $n^2 - 2n + 2$, and the proof of Lemma 1.4 shows that $L^0_{G,p}$ is conjugate either to $U_1 \times U_{n-1}$ in U_n or, for $n = 4$, to Sp_2 in U_4. In the latter case $L_{G,p}$ acts transitively on real directions in $T_p(M)$, and therefore, as earlier, we obtain that M is holomorphically equivalent to one of $\mathbb{B}^4$, $\mathbb{C}^4$, $\mathbb{CP}^4$. It can be shown that M in fact can only be equivalent to $\mathbb{C}^4$ and that an equivalence map can be chosen to transform G^0 into the subgroup $G_3(\mathbb{C}^4)$ that consists of all maps of the form (6.1) for $n = 4$ with $U \in Sp_2$. In the former case locally near p there are exactly two $I^0_{G,p}$-invariant germs of submanifolds of M corresponding to the 1- and $(n-1)$-dimensional $L^0_{G,p}$-invariant subspaces of $T_p(M)$, and, using this fact, we expect to be able to show that M is holomorphically equivalent to a product $M' \times S$, where $\dim_{\mathbb{C}} M' = n - 1$, and S is a Riemann surface. This structure should lead to product manifolds with M' being one of $\mathbb{B}^{n-1}$, $\mathbb{C}^{n-1}$, $\mathbb{CP}^{n-1}$, and S being one of Δ, $\mathbb{C}$, $\mathbb{CP}^1$.

Studying the actions of isotropy subgroups becomes harder as d_G decreases, but it still looks tractable. Closed connected subgroups of U_n of dimensions $(n-1)^2$ and $n^2 - 2n$ (corresponding to the cases $d_G = n^2 + 1$ and $d_G = n^2$) were described in Lemmas 2.1 and 4.2 of [IKru1], respectively. We also have an analogous description for $(n^2 - 2n - 1)$-dimensional subgroups of U_n corresponding to the case $d_G = n^2 - 1$, for $n \geq 3$.

We note that all complex manifolds of dimensions 2 and 3 homogeneous under Lie group actions were determined in [HL], [OR], [Wi], and thus in the study of the homogeneous case one can in fact assume that $n \geq 4$.

As we have seen in Chaps. 1–5, homogeneous manifolds make up only a small portion of our classification of hyperbolic manifolds. We do not expect to obtain a large number of G-homogeneous manifolds for general proper actions either. Most likely, the vast majority of manifolds will arise from the non-homogeneous case.

Due to Proposition 6.2, non-homogeneous manifolds can only occur for $d_G = n^2$ and $d_G = n^2 - 1$. It seems that in these cases one can completely classify all connected non-homogeneous manifolds of dimension $n \geq 2$ by appropriately modifying our arguments in Chaps. 3–5. For example, Proposition 3.2 generalizes as follows.

Proposition 6.3. *Let M be a connected non-homogeneous complex manifold of dimension $n \geq 2$. Suppose that $G \subset Aut(M)$ is a connected subgroup whose action on M is proper, and $d_G = n^2$. Fix $p \in M$ and let $V := T_p(O_G(p))$. Then*

(i) either the orbit $O_G(p)$ is a real or complex closed hypersurface in M, or p is a fixed point of the G-action;

(ii) if $O_G(p)$ is a real hypersurface, it is either spherical, or Levi-flat and foliated by manifolds holomorphically equivalent to one of $\mathbb{B}^{n-1}$, $\mathbb{C}^{n-1}$, $\mathbb{CP}^{n-1}$; there exist coordinates in $T_p(M)$ such that – with respect to the orthogonal decomposition $T_p(M) = (V \cap iV)^{\perp} \oplus (V \cap iV)$ – the group $L_{G,p}$ is either $\{id\} \times U_{n-1}$ or $\mathbb{Z}_2 \times U_{n-1}$, and the latter can only occur if $O_G(p)$ is Levi-flat;

(iii) if $O_G(p)$ is a complex hypersurface, it is holomorphically equivalent to one of $\mathbb{B}^{n-1}$, $\mathbb{C}^{n-1}$, $\mathbb{CP}^{n-1}$; there exist coordinates in $T_p(M)$ such that – with respect to the orthogonal decomposition $T_p(M) = (V \cap iV)^{\perp} \oplus (V \cap iV)$ – we have $L_{G,p} = U_1 \times U_{n-1}$, the subgroup $I'_{G,p} := \alpha_{G,p}^{-1}(U_1)$ is normal in G, and a biholomorphism between $O_G(p)$ and the corresponding manifold among $\mathbb{B}^{n-1}$, $\mathbb{C}^{n-1}$, $\mathbb{CP}^{n-1}$ can be chosen so that it transforms $G/I'_{G,p}$ (regarded as a subgroup of $Aut(O_G(p))$) into one of $Aut(\mathbb{B}^{n-1})$, $G\left(\mathbb{C}^{n-1}\right)$, $G\left(\mathbb{CP}^{n-1}\right)$, respectively; there are at most two complex hypersurface orbits in M;

(iv) if p is a fixed point of the G-action, then G is isomorphic to U_n, the manifold M is holomorphically equivalent to one of $\mathbb{B}^n$, $\mathbb{C}^n$, $\mathbb{CP}^n$, and an equivalence can be chosen so that it transforms G into U_n regarded in the standard way as a subgroup of one of $Aut(\mathbb{B}^n)$, $G(\mathbb{C}^n)$, $G(\mathbb{CP}^n)$, respectively.

Proof: As in the proof of Proposition 3.2, we consider the three cases, with differences occurring only in Cases 1 and 2.

In Case 1 we obtain, as before, that either $d = 0$ or $d = n - 1$. If $d = 0$, then $O_G(p) = \{p\}$, that is, p is a fixed point of the G-action. Then $G = I_{G,p}$ is compact, and Folgerung 1.10 of [Ka] implies that G is isomorphic to U_n, the manifold M is holomorphically equivalent to one of $\mathbb{B}^n$, $\mathbb{C}^n$, $\mathbb{CP}^n$, and an equivalence can be chosen so that it transforms G into U_n regarded in the standard way as a subgroup of one of $\mathrm{Aut}(\mathbb{B}^n)$, $G(\mathbb{C}^n)$, $G(\mathbb{CP}^n)$, respectively.

If $d = n-1$, then, as before, $O_G(p)$ is a complex closed hypersurface in M, and $L_{G,p} = U_1 \times U_{n-1}$, which implies that $L_{G,p}$ acts transitively on real directions in V. Thus, by Theorems 1.7 and 1.11, the orbit $O_G(p)$ is holomorphically equivalent to one of $\mathbb{B}^{n-1}$, $\mathbb{C}^{n-1}$, $\mathbb{CP}^{n-1}$. Next, the group $G/I'_{G,p}$ acts effectively on $O_G(p)$ and has dimension $n^2 - 1 = (n-1)^2 + 2(n-1)$. Then Theorem 6.1 yields that a biholomorphism between $O_G(p)$ and the corresponding manifold among $\mathbb{B}^{n-1}$, $\mathbb{C}^{n-1}$, $\mathbb{CP}^{n-1}$ can be chosen so that it transforms $G/I'_{G,p}$ (regarded as a subgroup of $\mathrm{Aut}(O_G(p))$) into one of $\mathrm{Aut}(\mathbb{B}^{n-1})$, $G\left(\mathbb{C}^{n-1}\right)$, $G\left(\mathbb{CP}^{n-1}\right)$, respectively.

In Case 2 we obtain, as before, that $O_G(p)$ is a closed real hypersurface in M, and that $O_G(p)$ is either strongly pseudoconvex (in which case $L_{G,p} = \{\mathrm{id}\} \times U_{n-1}$) or Levi-flat (in which case $L_{G,p} \subset \mathbb{Z}_2 \times U_{n-1}$). Supposing that $O_G(p)$ is Levi-flat and considering the leaf $O(p)_p$ of the corresponding foliation passing through p, we see that $L_{G,p}$ acts transitively on real directions in $T_p(O(p)_p)$ which gives that $O(p)_p$ is holomorphically equivalent to one of $\mathbb{B}^{n-1}$, $\mathbb{C}^{n-1}$, $\mathbb{CP}^{n-1}$. Supposing that $O_G(p)$ is strongly pseudoconvex, we obtain, as before, that it is spherical.

The proof is complete. ∎

Full generalizations of the classifications from Chaps. 3–5 will appear in our future work. We close the book by briefly outlining our classification results in two special cases, where G is isomorphic to either U_n (here $d_G = n^2$) or SU_n (here $d_G = n^2 - 1$); details can be found in [IKru1], [IKru2].

6.2 The Case $G \simeq U_n$

In this section we reproduce from [IKru1] our classification of connected complex manifolds of dimension $n \geq 2$ that admit an effective action of the group U_n by holomorphic transformations. According to Proposition 6.3, for a point $p \in M$ the orbit of p is one of the following: a single point, all of M (in this case M is compact), a real compact hypersurface in M, a complex compact hypersurface in M.

Manifolds that admit an action with a fixed point have been described in (iv) of Proposition 6.3. Taking into account that, up to inner automorphisms, every automorphism of U_n is either the identity or $g \mapsto \overline{g}$ for $g \in U_n$, we obtain the following slightly more precise statement.

Proposition 6.4. *Let M be a connected complex manifold of dimension n endowed with an effective action of U_n by holomorphic transformations that has a fixed point. Then M is holomorphically equivalent to one of $\mathbb{B}^n$, $\mathbb{C}^n$, $\mathbb{CP}^n$. A holomorphic equivalence f can be chosen to satisfy either the relation*

$$f(gq) = gf(q), \qquad (6.2)$$

or the relation

$$f(gq) = \overline{g}f(q), \tag{6.3}$$

for all $g \in U_n$ and $q \in M$.

Next, we consider manifolds homogeneous under a U_n-action. We start with an example of such a manifold. For $d \in \mathbb{C}^*$, $|d| \neq 1$, let $\mathfrak{M}_d$ be the *Hopf manifold* constructed by identifying every point $z \in \mathbb{C}^n \setminus \{0\}$ with $d \cdot z$, and let $[z]$ be the equivalence class of z. Choose a complex number λ such that $e^{\frac{2\pi(\lambda - i)}{nK}} = d$ for some $K \in \mathbb{Z} \setminus \{0\}$, and define an action of U_n on $\mathfrak{M}_d$ as follows. Represent every $g \in U_n$ as $g = e^{it} \cdot h$, where $t \in \mathbb{R}$, $h \in SU_n$, and set

$$g[z] := [e^{\lambda t} \cdot hz]. \tag{6.4}$$

We will now verify that this action is well-defined. Indeed, the same element $g \in U_n$ can be also represented in the form $g = e^{i(t + \frac{2\pi k}{n} + 2\pi l)} \cdot (e^{-\frac{2\pi i k}{n}} h)$, $0 \leq k \leq n-1$, $l \in \mathbb{Z}$. Then formula (6.4) yields

$$g[z] = [e^{\lambda(t + \frac{2\pi k}{n} + 2\pi l)} \cdot e^{-\frac{2\pi i k}{n}} hz] = [d^{kK + nKl} e^{\lambda t} \cdot hz] = [e^{\lambda t} \cdot hz],$$

as required. It is also clear that definition (6.4) does not depend on the choice of representative in the class $[z]$.

The action so defined is obviously transitive. It is also effective. For let $e^{it} \cdot h[z] = [z]$ for some $t \in \mathbb{R}$, $h \in SU_n$, and all $z \in \mathbb{C}^n \setminus \{0\}$. Then for some $k \in \mathbb{Z}$ we have $h = e^{\frac{2\pi i k}{n}} \cdot \mathrm{id}$, and for some $s \in \mathbb{Z}$ the following holds

$$e^{\lambda t} \cdot e^{\frac{2\pi i k}{n}} = d^s.$$

Using the definition of λ we obtain

$$t = \frac{2\pi s}{nK}, \quad e^{\frac{2\pi i k}{n}} = e^{-\frac{2\pi i s}{nK}}.$$

Hence $e^{it} \cdot h = \mathrm{id}$, and thus the action is effective.

Another example is provided by quotients of Hopf manifolds $\mathfrak{M}_d/\mathbb{Z}_m$ obtained from $\mathfrak{M}_d$ by identifying $[z]$ and $[e^{\frac{2\pi i}{m}} z]$, $m \in \mathbb{N}$. Let $\{[z]\} \in \mathfrak{M}_d/\mathbb{Z}_m$ be the equivalence class of $[z]$. We define an action of U_n on $\mathfrak{M}_d/\mathbb{Z}_m$ by the formula $g\{[z]\} := \{g[z]\}$ for $g \in U_n$. This action is clearly transitive; it is also effective, for example, if $(n, m) = 1$ and $(K, m) = 1$.

In fact, there are many effective transitive U_n-actions on quotients of Hopf manifolds, and it is possible to give a complete description of all such actions (see [I1]). Since there is no canonical transitive U_n-action in this situation, the following theorem deals with SU_n-equivariance rather than U_n-equivariance of the equivalence map.

Theorem 6.5. ([IKru1]) *Let M be a connected complex manifold of dimension $n \geq 2$ endowed with an effective transitive action of U_n by holomorphic* *transformations. Then M is holomorphically equivalent to $\mathfrak{M}_d/\mathbb{Z}_m$ for some $d \in \mathbb{C}^*$, $|d| \neq 1$, where $m \in \mathbb{N}$ and $(n,m) = 1$. A holomorphic equivalence $f : M \to \mathfrak{M}_d/\mathbb{Z}_m$ can be chosen to satisfy either relation (6.2) or, for $n \geq 3$, relation (6.3) for all $g \in SU_n$ and $q \in M$ (here $\mathfrak{M}_d/\mathbb{Z}_m$ is considered with the standard action of SU_n).*

We now turn to the case when all orbits of the U_n-action are real hypersurfaces. All such manifolds are classified in the following theorem.

Theorem 6.6. ([IKru1]) *Let M be a connected complex manifold of dimension $n \geq 2$ endowed with an effective action of U_n by holomorphic transformations. Suppose that all orbits of this action are real hypersurfaces. Then there exists $k \in \mathbb{Z}$ such that, for $m = |nk + 1|$, the manifold M is holomorphically equivalent to one of the following manifolds:*
(i) $S_r/\mathbb{Z}_m$ for some $0 \leq r < 1$ (see (2.1));
(ii) $S'_r/\mathbb{Z}_m$, where $S'_r := \{z \in \mathbb{C}^n : |z| > r\}$ and $r \in \{0, 1\}$;
(iii) $\mathfrak{M}_d/\mathbb{Z}_m$ for some $d \in \mathbb{C}^*$, $|d| \neq 1$.
A holomorphic equivalence f can be chosen to satisfy either the relation

$$f(gq) = \varphi^{-1}(g)f(q), \tag{6.5}$$

or the relation

$$f(gq) = \varphi^{-1}(\overline{g})f(q), \tag{6.6}$$

for all $g \in U_n$ and $q \in M$, where φ is the isomorphism

$$\varphi : U_n/\mathbb{Z}_m \to U_n, \qquad \varphi(g\mathbb{Z}_m) = (\det g)^k \cdot g, \qquad g \in U_n, \tag{6.7}$$

and $S_r/\mathbb{Z}_m$, $S'_r/\mathbb{Z}_m$, $\mathfrak{M}_d/\mathbb{Z}_m$ are equipped with the standard actions of $U_n/\mathbb{Z}_m$.

We note in passing that many "Hopf-like" manifolds that admit proper effective actions of n^2-dimensional groups for which every orbit is a real hypersurface, were ruled out as non-hyperbolic at step (IV) of the orbit gluing procedure in Sect. 3.4 (see the manifolds $M_{\hat{F}}$ there). All such manifolds must of course be included in a general classification for $d_G = n^2$.

We will now allow complex hypersurface orbits to be present in a manifold. Let $\widehat{\mathbb{B}^n}$, $\widehat{\mathbb{C}^n}$ and $\widehat{\mathbb{CP}^n}$ denote the blow-ups at the origin of $\mathbb{B}^n$, $\mathbb{C}^n$ and $\mathbb{CP}^n$, respectively (cf. (5.117)). Further, let $\tilde{S}'_r$ be the union of S'_r and the hypersurface at infinity in $\mathbb{CP}^n$. Each of $\widehat{\mathbb{B}^n}$, $\widehat{\mathbb{C}^n}$, $\tilde{S}'_r$ has one complex hypersurface orbit under the standard U_n-actions, and $\widehat{\mathbb{CP}^n}$ has two such orbits. We are now ready to formulate the final classification result of this section.

Theorem 6.7. ([IKru1]) *Let M be a connected complex manifold of dimension $n \geq 2$ endowed with an effective action of U_n by holomorphic transformations. Suppose that each orbit of this action is either a real or complex hypersurface and at least one orbit is a complex hypersurface. Then there*

exists $k \in \mathbb{Z}$ such that, for $m = |nk+1|$, the manifold M is holomorphically equivalent to one of the following manifolds:
(i) $\widehat{\mathbb{B}^n}/\mathbb{Z}_m$;
(ii) $\widehat{\mathbb{C}^n}/\mathbb{Z}_m$;
(iii) $\widehat{\mathbb{CP}^n}/\mathbb{Z}_m$;
(iv) $\tilde{S}'_r/\mathbb{Z}_m$, *where* $r \in \{0, 1\}$.
A holomorphic equivalence can be chosen to satisfy either relation (6.5) or relation (6.6) for all $g \in U_n$ and $q \in M$, where $\widehat{\mathbb{B}^n}/\mathbb{Z}_m$, $\widehat{\mathbb{C}^n}/\mathbb{Z}_m$, $\widehat{\mathbb{CP}^n}/\mathbb{Z}_m$, $\tilde{S}'_r/\mathbb{Z}_m$ are equipped with the standard actions of $U_n/\mathbb{Z}_m$.

Thus, Proposition 6.4 and Theorems 6.5, 6.6 and 6.7 give a complete classification of all complex manifolds of dimension $n \geq 2$ that admit effective actions of U_n by holomorphic transformations.

6.3 The Case $G \simeq SU_n$

In this section we reproduce a classification of connected complex manifolds of dimension $n \geq 2$ that admit effective actions of the group SU_n by holomorphic transformations (see [IKru2]). Certainly, all manifolds listed in Proposition 6.4 and Theorems 6.5, 6.6, 6.7 are part of this classification. However, as we will see below, there are manifolds that admit an action of SU_n without admitting an action of U_n.

First of all, it can be shown that if M is a connected complex manifold of dimension $n \geq 2$ endowed with an effective action of SU_n by holomorphic transformations, then for a point $p \in M$ the orbit of p is one of the following: a single point, a real compact hypersurface in M, a complex compact hypersurface in M (note that M cannot be homogeneous in this case).

Next, in the fixed point case one can prove the following analogue of Proposition 6.4.

Proposition 6.8. ([IKru2])*Let M be a connected complex manifold of dimension $n \geq 2$ endowed with an effective action of SU_n by holomorphic transformations that has a fixed point. Then M is holomorphically equivalent to one of $\mathbb{B}^n$, $\mathbb{C}^n$, $\mathbb{CP}^n$. A holomorphic equivalence can be chosen to satisfy either relation (6.2) or, for $n \geq 3$, relation (6.3) for all $g \in SU_n$ and $q \in M$.*

To formulate an analogue of Theorem 6.6, for $s \geq 1$ we set

$$E^{(4)}_{s,\infty} := \left\{(z, w) \in M^{(4)}_\mu : |z|^2 + |w|^2 > \sqrt{(s-1)/2}\right\}$$

(see **(7)(a)**). These manifolds are not hyperbolic and extend the family $E^{(4)}_{s,t}$ (see (5.30)) originally introduced for $1 \leq s < t < \infty$. Our next result is the following theorem.

Theorem 6.9. ([IKru2]) *Let M be a connected complex manifold of dimension $n \geq 2$ endowed with an effective action of SU_n by holomorphic transformations. Assume that all orbits of this action are real hypersurfaces. Then M is holomorphically equivalent to one of the following manifolds:*
(i) $S_r/\mathbb{Z}_m$ for some $0 \leq r < 1$;
(ii) $S'_r/\mathbb{Z}_m$ for some $r \in \{0, 1\}$;
(iii) $\mathfrak{M}_d/\mathbb{Z}_m$ for some $d \in \mathbb{C}^*$, $|d| \neq 1$;
(iv) $E^{(4)}_{s,t}$ for some $1 \leq s < t \leq \infty$ (here $n = 2$),
where $m \in \mathbb{N}$, $(n, m) = 1$. A holomorphic equivalence can be chosen to satisfy either relation (6.2) or, for $n \geq 3$, relation (6.3) for all $g \in SU_n$ and $q \in M$ (here $E^{(4)}_{s,t}$ is considered with the standard action of SU_2).

Finally, we allow complex hypersurface orbits to be present in the manifold. Recall that the manifold $M^{(4)}_\mu$ introduced in **(7)(a)** is in fact $\mathbb{C}^2 \setminus \{0\}$ with a non-standard complex structure obtained by pull-back under the map Φ_μ (see (5.29)). It is straightforward to show from the explicit form of Φ_μ that the complex structure of $M^{(4)}_\mu$ extends to a complex structure on $\mathbb{CP}^2 \setminus \{0\}$. We denote $\mathbb{CP}^2 \setminus \{0\}$ with this extended complex structure by $\tilde{M}^{(4)}_\mu$. Let $\mathcal{O}_{10}$ be the complex curve at infinity in $\mathbb{CP}^2$ (clearly, $\tilde{M}^{(4)}_\mu = M^{(4)}_\mu \cup \mathcal{O}_{10}$). In the complex structure of $\tilde{M}^{(4)}_\mu$ the set $\mathcal{O}_{10}$ is a complex curve whose complex structure is identical to that induced from $\mathbb{CP}^2$. The action of SU_2 on $M^{(4)}_\mu$ extends to an action by holomorphic transformations on all of $\tilde{M}^{(4)}_\mu$, and $\mathcal{O}_{10}$ is an orbit of this action. For $s \geq 1$ we now define

$$\tilde{E}^{(4)}_s := E^{(4)}_{s,\infty} \cup \mathcal{O}_{10}.$$

Our final classification result is the following theorem.

Theorem 6.10. ([IKru2])*Let M be a connected complex manifold of dimension $n \geq 2$ endowed with an effective action of SU_n by holomorphic transformations. Suppose that each orbit of this action is either a real or complex hypersurface and at least one orbit is a complex hypersurface. Then M is holomorphically equivalent to one of the following manifolds:*
(i) $\widehat{\mathbb{B}^n}/\mathbb{Z}_m$;
(ii) $\widehat{\mathbb{C}^n}/\mathbb{Z}_m$;
(iii) $\widehat{\mathbb{CP}^n}/\mathbb{Z}_m$;
(iv) $\tilde{S}'_r/\mathbb{Z}_m$ for some $r \in \{0, 1\}$;
(v) $\tilde{E}^{(4)}_s$ for some $s \geq 1$ (here $n = 2$),
where $m \in \mathbb{N}$, $(n, m) = 1$. A holomorphic equivalence can be chosen to satisfy either relation (6.2) or, for $n \geq 3$, relation (6.3) for all $g \in SU_n$ and $q \in M$.

Thus, Proposition 6.8 and Theorems 6.9, 6.10 give a complete classification of all complex manifolds of dimension $n \geq 2$ that admit effective actions of SU_n by holomorphic transformations. Among the manifolds listed there only $E^{(4)}_{s,t}$ and $\tilde{E}^{(4)}_s$ do not admit an action of the unitary group.

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